Quantization and Coding for Decentralized LTI Systems[1]

Serdar Yüksel[2] and Tamer Başar[2]
(yuksel, tbasar)@control.csl.uiuc.edu

***Abstract*— We study the communication rate requirements for centralized and decentralized control schemes when the plant and the controller are connected via a noiseless bandlimited channel. We introduce recursive quantizers that achieve monotonic boundedness and exponential stability of the worst-case state estimation error with minimum rate. Rate requirements for centralized schemes are shown to be lower than those for decentralized schemes. A quantification of the information sharing between the controllers, such as full, instant, and one-step delayed information sharing, is shown to crucial for communication requirements and complexity. Slepian-Wolf coding argument is used to show that information sharing by the controllers, and not by the plants, is sufficient to achieve the lower bound on the rate, and schemes confirming this efficiency are constructed. It is also shown that delay in communication between the controllers leads to higher rate requirements between the controllers and the plants.**

I. Introduction

Control of physically distant systems has emerged as a major research topic in recent years. Such systems require a joint analysis of communications and control for system design and optimality, since effects of communication constraints such as finite precision, delay, bandwidth consumption and decentralization might affect the system characteristics dramatically. A diverse range of structures and models have been analyzed by various authors, with a selected incomplete list of such work being [2], [3], [4], [5], [6], [7], [8], [9]. All these address issues in the context of centralized systems, with decentralized systems not having attracted much attention heretofore. Our goal here is to study such systems from the point of view of bandwidth limitation and its effect on control performance under various information structures.

The organization of the paper is as follows. We first introduce, in section II, the class of systems and the issues to be studied, and provide a brief overview of quantization, before considering, in section III, the centralized case. In section IV, we extend the analysis to decentralized systems and consider coding and decoding schemes for LTI systems with different information sharing structures involving the plants and the controllers, which are connected via a band-limited channel. The paper ends with simulations presented in section V. Due to page limitations, all details of the derivations could not be included here; they can be found in the full paper or [1].

II. Problem Description

We consider discrete-time LTI systems of the form

$$x_{t+1} = Ax_t + Bu_t\,,\ t \geq 0, \qquad (1)$$

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[2]Coordinated Science Lab., Univ. of Illinois, Urbana, IL 61801-2307 USA.

where A is an $n\times n$ matrix, B is an $n\times m$ matrix, and u is an $m-$dimensional control vector. For ease of presentation, we assume throughout that the matrix A has only real eigenvalues. The initial state vector x_0 is assumed to be the realization of a random vector X_0 with a finite support set. The controller and the plant are separated by a bandlimited communication line, and the controller generates the control signal u based on quantized information it receives on x, under some information structure to be delineated later. The objective is to design optimal dynamic quantizers, $Q := \{Q_t(X),\ t = 0, 1, 2, \ldots\}$ and encoder-decoder pairs under some stability criteria. Before introducing these criteria, we first recall the notions of a quantizer, and *support width* of a random variable.

Definition II.1. *[10] A quantizer $Q(x)$ is a function that maps a large, possibly infinite, set (where a variable x takes values) into a smaller finite set (where the quantized values lie), and is characterized by a set of thresholds or bin edges that partition the input space into quantization bins and a set of reconstruction values as the representative values of the corresponding bins. The class of all quantizers compatible with the underlying information structure is denoted by $\mathcal{Q}$.*

Definition II.2. *For a scalar (one-dimensional) random variable , the* support width, *W, is the width of the domain over which the probability density function is non-zero:*

$$W(X) = \int 1_{(f(x)\neq 0)} dx.$$

For a multi-dimensional random variable, the support width is the sum of the support widths corresponding to each of the marginal densities.

The stability criteria for selection of appropriate quantizers are now given below. Throughout, we let $\hat{x}_t$ denote the least-squares estimate of x_t at the receiver (controller) based on all the quantized signals available at the receiver up to (and including) time t, and let $e_t := x_t - \hat{x}_t$ denote the deviation of this estimate from the true value of x_t; to economize notation, we will sometimes drop the subscript 't'.

Stability Criteria: Let there exist a causal time-invariant quantizer Q under which, for some $a \in (0, 1)$, $W(e_{t+1}) \leq aW(e_t)\quad \forall t \geq 0$. Then we have *exponential stability in uncertainty*. If the same holds with only $a = 1$, then we have *monotonic boundedness in uncertainty*.

Our objective is to construct quantizers which will lead to either exponential stability or monotonic boundedness. If the error e is uniformly distributed, then the above will imply monotonic boundedness and stability, respectively, in the entropy of the state estimation error. Note that a *forward-looking* criterion [2] is also possible, which is not considered here.

III. Centralized Scheme

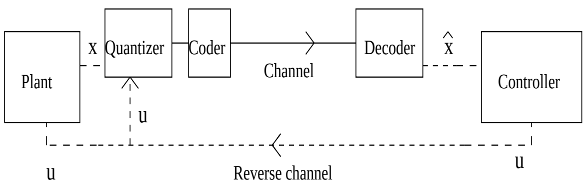

Fig. 1. System structure.

In centralized schemes, there is a single plant with a single controller, and hence in essence the system is a single multi-dimensional system (Fig. 1). To design such a scheme with digital noiseless channels, one needs to investigate *vector quantization*, which is a slight generalization of scalar quantization, where the quantization bins and reconstruction values belong to higher dimensions (see [10]). Vector quantization has a higher degree of freedom over scalar quantization of vector components, since for the quantization the components can have a joint density which does not have to be of product form. For the implementation, since the analytic derivations for the optimal quantizer design are difficult to obtain, the optimization is performed over training sequences which are basically the samples generated according to the underlying stochastic distribution [13]. In general, the advantage of vector quantization over scalar quantization of vector components diminishes as the statistical dependence between the vector components decreases. A scheme to achieve independence would be to apply a mapping by conditioning, so as to have statistically independent components. This method is appropriate for control systems, which can be converted to a diagonal or a lower triangular form where there is a causality in the evolution of the vector state components, and the conditioning on the previous components will yield a scalar distribution per each component. We will thus use a *sequential scalar quantization* of vector components in our analysis. We assume that the system matrix is diagonalizable, which will lead to independent evolution for each component.

The motivation for this is that for a scalar system, the quantization error converges to a random variable with a uniform pdf, as stated in the following lemma [1], [2].

Lemma III.1. *Let x_0 be the realization of a random variable X_0 with a continuous probability density function $f_0(.)$ and with finite support $[-|a|\Delta, |a|\Delta]$, and suppose that at each time step, $x_{t+1} = ax_t$ is quantized successively using an $|a|$ level uniform quantizer, where $|a|$ is a non-zero integer. Then, the Kullback-Leibler distance between the uniform error density and conditional quantization error density (the quantization error for a specific bin) converges to zero as $t \to \infty$.*

We now consider the general n-dimensional system.

Theorem III.2. *Consider the n-dimensional system $x_{t+1} = Ax_t + Bu_t$, with A being diagonalizable, with real eigenvalues $\lambda_i, \;\; i = 1, \ldots, n$. If one applies sequential time-invariant scalar quantization, the rate required for the boundedness of the worst-case estimation error is at least $\sum_i \max(0, \log_2(|\lambda_i|))$. For exponential stability, the rate required is strictly larger than this amount.*

Proof. First we will use an information theoretic approach to obtain the ultimate lower bound and then use a quantization technique to show that this bound is achievable. A can be written in the form $A = \Pi\Lambda\Pi^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. If the components are made statistically independent by conditioning along with the coordinate transformation, the rate required will be the sum of the rates required for each scalar subsystem. Using Lemma III.1, the distribution for the error will be uniform after a sufficiently large number of steps. In this analysis, since the control functions are transferred with no loss, they do not contribute to the evolution of the uncertainty. For any scalar subsystem, for any stage i, the rate is lower bounded by the minimum mutual information between the state and its quantized value:

$$\begin{aligned} R &\geq \min_{Q \in \mathcal{Q}} I(\hat{x}_i | \hat{x}_0^{i-1}; x_i | \hat{x}_0^{i-1}) \\ &= H(x_i | \hat{x}_0^{i-1}) - H(x_i | \hat{x}_i, \hat{x}_0^{i-1}), \end{aligned} \tag{2}$$

where $\hat{x}_0^{i-1}$ denotes the past information of the quantizer outputs, which are available both at the transmitter and the receiver. Since the system is Markov, the entire information that can be inferred from the past is captured by the latest quantization outcome [11], and is useful in the construction of the encoder and decoder since it is available at both sites. Thus (2) becomes

$$\begin{aligned} R &\geq \min_{Q \in \mathcal{Q}} I(\hat{x}_i | \hat{x}_{i-1}; x_i | \hat{x}_{i-1}) \\ &= H(x_i | \hat{x}_{i-1}) - H(x_i | \hat{x}_i, \hat{x}_{i-1})\,, \end{aligned} \tag{3}$$

and since $x_i = \lambda_i x_{i-1}$,

$$H(x_i - \hat{x}_i | \hat{x}_i) \geq \log_2(|\lambda_i|) + H(x_{i-1} - \hat{x}_{i-1} | \hat{x}_{i-1}) - R\,.$$

Therefore, we need to have at least an additional $\log_2(|\lambda_i|)$ bits per stage to make the entropy nonincreasing.

Using a quantization theoretic argument, let the ith eigenvalue be λ_i, and let the associated estimation error have a support width of Δ_i. After one time step, the support width will be $|\lambda_i|\Delta_i$. If a time-invariant K_i-level uniform quantizer is used at each stage, the support width in the next stage becomes $\frac{|\lambda_i|}{K_i}\Delta$, which will be stable only when $K_i \geq |\lambda_i|$. If one uses fixed code length for each of the quantization bins, with a length of $\log_2(K_i)$, the minimum rate required for boundedness is $\log_2(|\lambda_i|)$. Thus, the information theoretic bound is indeed achievable by a single step quantization approach, if the eigenvalue is an integer. If λ_i is not an integer, a rate better than $\log_2(\lceil|\lambda_i|\rceil)$ can be achieved using the distortion constrained entropy minimizing quantizer introduced in [2].

Sufficiency and construction. The error in the estimation of the state will be quantized and fed into the channel, as depicted in Fig. 2. Let λ_i be the eigenvalue of the ith subsystem after diagonalization, whose state is denoted by x^i.

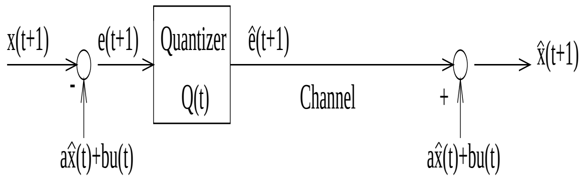

Fig. 2. The quantization scheme; a represents λ_i.

Then, the scalar system equation for the ith subsystem is $x^i_{t+1} = \lambda_i x^i_t + b^i u_t$, where b^i is a row vector. Let $\hat{x}^i_t$ denote the estimate for x^i_t using quantized information. Define the error e as

$$e^i_{t+1} := x^i_{t+1} - \lambda_i \hat{x}^i_t - b^i u_t. \quad (4)$$

Let $\hat{e}^i$ be the quantization outcome of the error, e^i. Then, at the receiver we have; $\hat{x}^i_{t+1} = \lambda_i \hat{x}^i_t + b^i u_t + \hat{e}^i_{t+1}$. These lead to: $e^i_{t+1} = \lambda_i(x^i_t - \hat{x}^i_t)$. Also, we have $x^i_t - \hat{x}^i_t = e^i_t - \hat{e}^i_t$.

Thus, we reduce the error on the state to the error in the estimation. Finally, the following recursion follows:

$$e^i_{t+1} = \lambda_i(e^i_t - \hat{e}^i_t). \quad (5)$$

Thus, the error introduced by the quantization becomes the signal to be quantized at the next stage. To ensure the estimation error to be a nonincreasing sequence, it suffices to enforce $E[(e^i_t)^2] \le E[(e^i_{t-1})^2]$, which corresponds to

$$\frac{E[(e^i_t)^2]}{E[(e^i_t - \hat{e}^i_t)^2]} \ge \lambda_i^2. \quad (6)$$

The proof is now completed with the following:

Lemma III.3. *In an LTI system with a uniformly distributed initial state, for uniform quantization to ensure that the estimation error variance is a nonincreasing sequence with respect to time, the rate $R \ge \sum_i \max(0, \log_2(|\lambda_i|))$ bits per sample.*

Proof. Assume that the current support of the uniform pdf is $[-\Delta, \Delta]$. If we use a K-level quantizer, the variance in the quantization error will be $(1/3)(\Delta/K)^2$, and the variance of the error at the next stage will be $(\lambda_i^2/3)(\Delta/K)^2$ and from (5) we want this to be smaller than $\Delta^2/3$, which means that we should have $|\lambda_i| \le K$. We therefore need a quantizer with the number of levels at least as large as $|\lambda_i|$, and to achieve a given K number of levels, we know that we need $R = \log_2(K)$ (since the symbols will be uniformly distributed the bit rate is identical to the entropy of the random variable). Hence,

$$R \ge \sum_i \max(0, \log_2(|\lambda_i|)). \quad (7)$$

Corollary III.4. *Given a uniformly distributed input, for any recursive time-invariant quantizer to achieve exponential stability of the state estimation error variance, the rate has to be strictly greater than $\sum_i \max(0, \log_2 |\lambda_i|)$. This is achievable by any K_i level uniform quantizer with $K_i > |\lambda_i|$.*

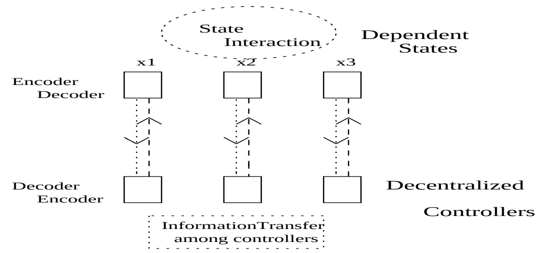

Fig. 3. Decentralized structure

Remark. Rate-distortion theory provides the ultimate bound for the rate required to achieve a given level of distortion [12]. In most cases, the rate distortion bound is not achievable by a single step quantization and requires long block codes, which is not appropriate for control since it could entail significant delays. Nonetheless, we find the bound obtained by such an analysis useful, as the following shows [1].

Proposition III.5. *For a linear system $x_{t+1} = Ax_t$, where the initial state x_0 is uniformly distributed and the type of quantization is uniform, the bit rate required for monotonic boundedness of the state estimation error entropy is $\max(0, \sum_i \log_2(|\lambda_i|))$, where λ_i's are the eigenvalues of A.*

IV. Decentralized Schemes

For the decentralized case, we consider an LTI system of form in (1), where $B = \text{diag}(b_1, \dots, b_n)$, A is an $n \times n$ diagonalizable matrix with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, x_t is an n-dimensional state vector, and u is an n-dimensional control vector. Since B is diagonal, all the control terms have direct impact on the associated state with the same index i, $i \le n$. Thus the system is decentralized in the sense that each controller state pair, a subsystem, is decoupled from the other ones as is depicted in Fig. 3. Thus, for instance, for the ith component (x^i) of the state, the evolution will be:

$$x^i_{t+1} = a_{i1}x^1_t + a_{i2}x^2_t + \dots + a_{in}x^n_t + b_i u^i_t \quad (8)$$

In case there are no communication and quantization effects, u^i_t will be a function of the perfect state information available to it. If the controller has access to all the observations corresponding to other state components and the control functions, then the system will effectively be a centralized one. In contrast, we consider in this section the situation where each controller does not have access to the control functions used by the remaining controllers, and we investigate different cases where various different types of information of state and control are available through the channels to the controllers, which is related to the amount of communication among the controllers. The primary interest is to introduce a rate analysis for reliable state estimation.

A. Plants and controllers share the state estimates

In this scheme, the estimated states, but not the control signals, are shared between the controllers. As a further assumption, the encoder is assumed to have access to the corresponding receiver's information. In this case if we condition on the immediate past, due to the Markov property of the state, the further past data will be of no use [11]. Let us consider the first component: If we subtract the predicted value

$$a_{11}\hat{x}^1 + a_{12}\hat{x}^2 + \ldots + a_{1n}\hat{x}^n + b_1 u^1,$$

from the actual state x_{t+1}, the innovation to be sent will be the sum of the individual quantization errors, which are independent given the past state estimates.

Here an asymptotic analysis is possible, as well as a recursive one. Even if the initial distributions are uniform for each component, the quantization error at a later time will not be uniform due to the additive nature in the noise terms, whose probability density computation requires a series of convolution operations. Note that the main difference between the centralized scheme and this scheme lies in the unavailability of the control signals. If those were also known, then the system could be transformed into a diagonal form, which would lead to the analysis done in the centralized case.

Proposition IV.1. *Assume that the initial state of each subsystem has a finite support pdf. If fixed length coding is used, the rate required for exponential stability of the worst-case error is strictly larger than* $\sum_i \max(0, \log_2(|\lambda_i|))$. *To achieve boundedness, the rate required is lower bounded by the same.*

Proof. Since the information structure above is inferior to the sequential scalar quantization case discussed earlier, the rate required for that case can be regarded as a lower bound for the scheme under consideration for this case. Note that in the decentralized scheme, sequential analysis is inevitable, since a joint analysis is not possible. $\diamond$

The above analysis is based on an information theoretic argument. Now, we restrict the analysis to time-invariant quantizers with fixed length codes, where the signal to be coded will be the state conditioned on the information set.

Proposition IV.2. *To achieve boundedness of the quantization error with a time-invariant quantizer, with fixed length codewords, the minimum rate required is given by the solution of the following optimization problem: Let* A^+ *be obtained by replacing all entries of* A *with their absolute values, and let* $\mathcal{K}$ *denote the class of diagonal* $n \times n$ *matrices* K *(with the* i*th diagonal element denoted by* K_{ii}*), where* $\frac{1}{K_{ii}}$ *are integers, and further* KA^+ *is stable. Then, the optimization problem is:*

$$R = \min_{\mathcal{K}} \sum_i \log_2(1/|K_{ii}|). \tag{9}$$

Proof. Due to Markovness of the system, for the ith component, the signal to be quantized is the conditioned state on the quantized data of the previous state:

$$e_{t+1} = (x^i_{t+1} | \hat{x}^1_t, \hat{x}^2_t, \ldots, \hat{x}^n, bu^1_t),$$

which is identical to

$$a_{i1}(x^1_t - \hat{x}^1_t) + a_{i2}(x^2_t - \hat{x}^2_t) + \ldots + a_{in}(x^n_t - \hat{x}^n_t). \tag{10}$$

Defining the quantization error for each component: $q^i_t := (x^i_t - \hat{x}^i_t)$, we have $e^i_{t+1} = a_{i1}q^1_t + a_{i2}q^2_t + \ldots + a_{in}q^n_t$, which in matrix form is $e_{t+1} = Aq_t$.

Assuming that the initial condition for each component x^i_0 to be bounded, not necessarily being uniform, at time t before quantization, the support width W of the random variable to be quantized, e^i, will be $|a_{i1}|\Delta^1_t + \ldots + |a_{in}|\Delta^n_t$. If one uses a K_{ii} level uniform quantization, the support width of each bin will be $\big(|a_{i1}|\Delta^1_t + \ldots + |a_{in}|\Delta^n_t\big)/K_{ii}$, and using this approach for each dimension: $\Delta_{t+1} = KA^+\Delta_t$.

Thus, the problem is reduced to a linear system problem, and the stability of the estimation error can be investigated by finding the eigenvalues of the matrix KA^+, and requiring them all to be in the unit circle. The rate required is the sum of the logarithms of the numbers of levels for each subcomponent in the system, and is thus equal to $\sum_i \log_2(\max(1, \frac{1}{K_{ii}}))$. $\diamond$

In case no information is transmitted for a particular subcomponent of the control system, the number of levels required can be taken to be 1, since the rate required will be zero. Therefore the matrix K has elements at most equal to 1 on the diagonal. Thus, an optimization problem can be posed as:

$$\min_{\mathcal{K}} \sum_i \log_2(1/|K_{ii}|), \tag{11}$$

which is the minimum achievable rate. But since $\sum_i \log_2(\frac{1}{|K_{ii}|}) = \log_2 \prod \frac{1}{|K_{ii}|}$, logarithm is a monotonic function, and K is diagonal, the problem reduces to one of finding the optimal K_0 matrix solving $\det(K_0) = \max_{\mathcal{K}} \det(K)$.

Proposition IV.3. *When* A *is a diagonal matrix, the minimum fixed code length data rate as the solution of (11) is* $R = \sum_i \log_2(C(|\lambda_i|))$, *where* $C(x)$ *is the smallest integer that is strictly larger than* x *(and is thus a modified* ceiling *function).*

Proof. When A is diagonal, the condition for stability is identical to: $\frac{1}{|K_{ii}|} > |\lambda_i|$. To achieve strict inequality using the constraint that $1/K_{ii}$ is an integer: If λ_i is not an integer; $\frac{1}{K_{ii}} = \lceil(|\lambda_i|\rceil)$, whereas if λ_i is an integer, $\frac{1}{K_{ii}} = |\lambda_i| + 1$. Thus, the rate is: $R = \sum_i \log_2(C(|\lambda_i|))$. $\diamond$

Remark. If the pdf is uniform, a lower rate can be achieved using the distortion-constrained entropy minimizing quantizer [2], which exploits the advantage of variable length coding. $\diamond$

Proposition IV.4. *The optimization problem (9) admits a solution for any diagonalizable matrix* A.

Proof. If one uses $\big(1/K_{ii}\big) = C(\max_i |\lambda_i|) =: |\lambda_m|$, the matrix KA^+ can be expressed as:

$$KA^+ = KS^{-1}\Lambda S = \frac{1}{|\lambda_m|}S^{-1}\Lambda S = S^{-1}\frac{\Lambda}{|\lambda_m|}S,$$

which has all its eigenvalues in the unit circle. The rate in this case becomes: $R = n\log_2(|\lambda_m|)$. There are only finitely many choices for the matrix K which are better than the one proposed, and there should exist at least one solution. $\diamond$

B. Only the controllers share the state estimates

We now consider a structure where the plants do not communicate with each other, and where the controllers share their data regarding estimation. Since each is able to compute the necessary control function after receiving the estimates from the other controllers, the controllers do not have to share the control functions. We now state the following:

Proposition IV.5. *To achieve boundedness of the quantization error with a time-invariant quantizer, with fixed length codewords, the minimum rate required is given by the solution of the problem introduced in Proposition IV.2, and thus there is no loss of performance when compared with the case where the subplants have access to the estimates at the controllers.*

Proof. The conceptual proof is based on the Slepian-Wolf coding theorem [14]. Slepian and Wolf showed that if some random data are not available at the encoder, but are available at the decoder, then the data rate can still be decreased by conditioning on the information available at the decoder.

Consider a 3-dimensional system. Then, the Slepian-Wolf theorem says that (also using the Markov property) the rate required for each channel is bounded from below by:

$$R \geq H(x^1_{t+1} | \hat{x}^1_t, \hat{x}^2_t, \hat{x}^3_t)\,. \tag{12}$$

Thus, in theory, there is no loss in optimality, and the communication between the controller and the plant with regard to the state information is unnecessary. Below we show that this is indeed the case in practice as well.

Construction. Obtaining constructions that achieve Slepian-Wolf coding efficiency remains a difficult and open problem. Schemes that use long block codes and codes with memory are available, but these are mostly impractical for control systems because of the significant delay they entail. There are approaches based on binning and coset formation ([15], [16]) that are more practical but mostly optimal only in the asymptotic sense. However, for the problem we consider, Slepian-Wolf gain is indeed achievable with a single step quantization using a binning-type argument. The approach we take is based on a uniform quantization interpretation and exploits the optimality based on the assumption that the quantization and estimation errors are bounded.

The data to be sent, x^i_{t+1}, can be written as $x^i_{t+1} = a_i \hat{x}_t + a_i q_t$, where $\hat{x}_t$ is the vector of the state estimates available at each subcontroller, q_t is the vector of quantization error at time t, and a_i is the ith row of matrix A^+. In this problem the side information available is the term $\hat{x}_t$.

The goal again is to have a nonincreasing sequence for the support width of the estimation error density. Assume that d^i_t is the support width of the uncertainty of x^i_t, and suppose that d^i_{qt} is the support width of the random variable $a_i q_t$. Since the support width d^i_{qt} is to be finite, the side information and the state differ at most by the width of the support, and this will be the idea behind the construction scheme. As seen from Fig. 4, there are n cosets, where n is 4 in the figure. The quantizer therefore needs only inform the receiver which coset the data

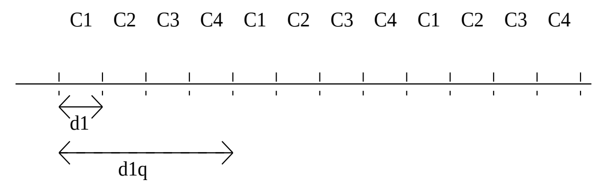

Fig. 4. The decoder can decode the signal by just knowing the coset.

belongs to. The receiver will use the coset and decode the signal using the available side information. The rate required will then be $\log_2(d^i_q/d^i)$.

Let us assume that the encoder sent $C1$. In this case, the receiver will just need to choose between the possible bins in the coset. Since it is assumed that there is always a nearest one without ambiguity, the evolution of the vector d_q will be $(d_q)_{t+1} = A^+ d_t$. After quantization, $d_{t+1} = (1/K_i) d_{qt}$, which is equivalent to $d_{t+1} = (1/K_i) A^+ d_t$. Now, define K as the diagonal matrix consisting of the reciprocals of the number of quantization levels on its diagonals. Recognizing this expression from the previous section, an optimization problem can be proposed as in (11). ⋄

C. Information sharing is delayed

We now consider a different scheme where information on the stage before the last one, including the control values, is available. Consider the first subsytem. Before quantization

$$q^1_{t+1} = a_{11} q^1_t + a_{12} \sum_i a_{2i} q^i_{t-1} + a_{13} \sum_i a_{3i} q^i_{t-1} + \dots$$

Thus, in vector form, after quantization, we have:

$$q_{t+1} = K((A - \text{diag}(A))A) q_{t-1} + K \text{diag}(A) q_t. \tag{13}$$

Now, for the worst case estimation error, we need to consider the absolute entries of the above matrices. Defining $N := (A - \text{diag}(A))A$, for the support size W of the estimation errors, we have

$$W_{t+1} = K(N^+) W_{t-1} + K \text{diag}(A^+) W_t. \tag{14}$$

Let $\mathcal{K}_n$ denote the set of diagonal positive and real matrices K satisfying stability condition in system (14). The second-order system in (14) can be reduced to a first-order system in the usual way: Define $W_{t-1} = r_t,\ W_t = p_t$. Then (14) becomes:

$$\begin{bmatrix} r_{t+1} \\ p_{t+1} \end{bmatrix} = B \begin{bmatrix} r_t \\ p_t \end{bmatrix}, \quad B := \begin{bmatrix} 0 & I \\ KN^+ & K(\text{diag}(A^+_{ii})). \end{bmatrix}$$

The optimization problem is to find a K_0 such that

$$\det(K_0) = \max_{\mathcal{K}} \det(K). \tag{15}$$

Proposition IV.6. *When A is diagonal, (15) is equal to*

$$R = \sum_i \log_2(C(|\lambda_i|)).$$

Proof. Identical to the full information-sharing case. ⋄

Remark. (14) has fewer requirements than the scheme where the controller ignores its very last estimate of its associated state information, which would lead to $q_{t+1} = K[A^2]^+ q_{t-1}$. This rate could also be used as an upper bound on the rate requirements, which would yield a rate of $n \log_2(|\lambda_{max}|)$, where λ_{max} is the maximum eigenvalue of $[A^2]^+$. $\diamond$

Hence, there is a trade-off: as delay in communication between the decision makers increases the rate required between the plants and the decision makers increases.

V. SIMULATIONS

A. Centralized scheme:

We consider a centralized system with a random initial state vector, where each component is i.i.d., uniformly distributed on $[-1, 1]$. The system equation is as in (1), where

$$A = \begin{bmatrix} -4 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Since controls have no effect on the evolution of uncertainty, they will be taken to be zero in the simulations without any loss of generality. In Fig. 5, stability of the state estimation error is illustrated. The rate used is $\sum_i \log_2(\lceil |\lambda_i| \rceil) = 6.49$ bits/stage, and stability in estimation error is achieved. The results confirm the analytical results, since the rate used is greater than the minimum required rate in (7).

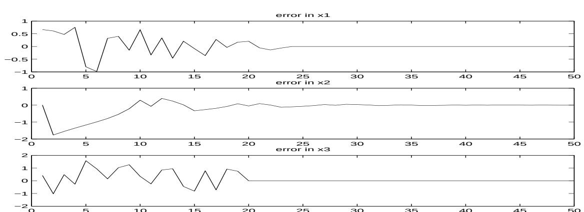

Fig. 5. Centralized scheme

B. Decentralized Schemes

1) Side information on estimates is available: Same system as above is used for the simulation in a decentralized setting. The rate $R = 6.49$ bits/stage, which was a stabilizing rate for the centralized scheme, is used and an unstable output is obtained (Fig. 6). However, if R is increased to say

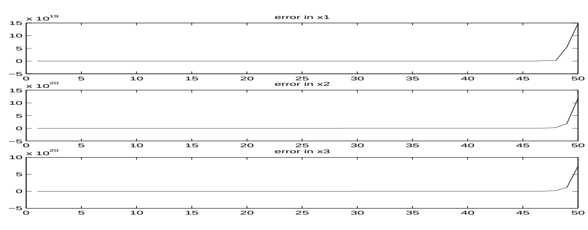

Fig. 6. Decentralized unstable system

$R = 8.4221$ bits/stage, stability is achieved (Fig. 7). Here the number of levels is the ceiling function of the eigenvalue of A^+ with the maximum absolute value, which is 7.

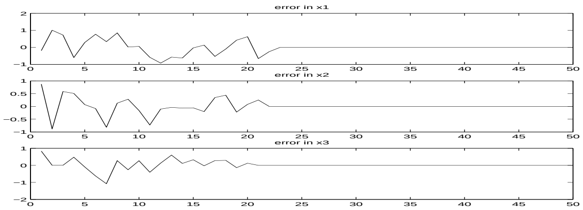

Fig. 7. Decentralized stable system

2) No side information at the plant: We consider a scalar system, where the plant does not have access to the side information at the receiver. The state evolves as $x_{t+1} = 1.5x_t + bu_t$. The rate used in the simulation is 2 bits/stage. A time-invariant quantizer is used, and Slepian-Wolf efficiency and stability is achieved (Fig. 8).

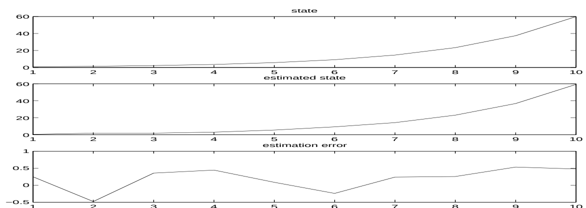

Fig. 8. Stability in estimation with no estimator information at the plant

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