

University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2000

Problem Set 7

Hypothesis Testing, Detection and Estimation

Issued: Thursday, October 19th. **Due:** Thursday, October 26th (beginning of lecture).

Reading from Lathi: Chapter 10.

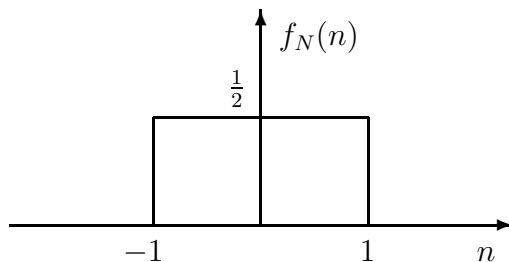
Reading from Haykin: Chapter 4, Sections 4.1–4.5.

Problem 7.1 (Optional)

Problems 10.5-2 and 10.5-3 from Lathi, p. 486.

Problem 7.2

Consider the following communication scenario: under hypothesis H_0 no signal is transmitted and the receiver observes a (constant) signal $R = N$, whereas under hypothesis H_1 a constant signal is transmitted and the receiver observes $R = 1 + N$. The random variable N models noise in the channel and is assumed to have the pdf shown below.



The prior probabilities for hypotheses H_0 and H_1 are given by

$$\Pr(H_0) = 1/4, \quad \Pr(H_1) = 3/4.$$

As we discussed in class, given a received signal $R = r$, we can use the MAP rule at the receiving end in order to minimize the probability of error. Furthermore, we know that the MAP rule reduces to the likelihood ratio test

$$\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{H_0}{\overset{H_1}{>}} \frac{\Pr(H_0)}{\Pr(H_1)}.$$

- (a) Find the range or ranges of values of the observation r for which you would decide that the signal was transmitted (“ H_1 ” has taken place).
- (b) Find the corresponding (minimal) probability of error.

Problem 7.3

Consider a system for determining whether a certain communication channel is being used or not. Let H_1 denote the hypothesis that the channel is being used, and let H_0 denote the hypothesis that the channel is not being used. The decision between hypotheses H_1 and H_0 is based on a single scalar measurement R (e.g., the output of our antenna at a particular time). This measurement R is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 : f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 : f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} .$$

Assume that the a priori probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

- (a) Find the decision rule that minimizes the probability of error.
- (b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 ,$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma_1$ and γ_2 are appropriate constants, and $Q(\gamma)$ is defined as

$$Q(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau .$$

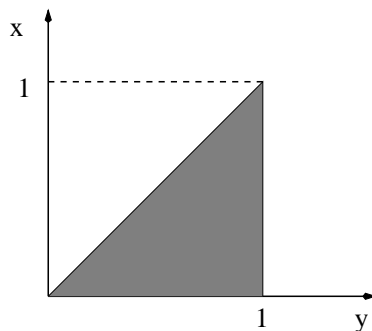
- (c) Is there a choice of p_0 and p_1 such that the rule of part (b) always decides in favor of one of the hypothesis regardless of the measurement R ?

Problem 7.4

A signal X which has Gaussian distribution with mean $\mu_X = 2$ and variance $\sigma_X^2 = 0.5$ is sent via a cable. The receiver needs to form an estimate \hat{X} for the transmitted signal X based on the received signal $Y = X + N$, where N models the (*additive*) noise. Assume that N also has Gaussian distribution with mean $\mu_N = 0$ and variance $\sigma_N^2 = 2$. Find \hat{X}_{MMSE} , the minimum mean square error estimator for X given Y .

Problem 7.5

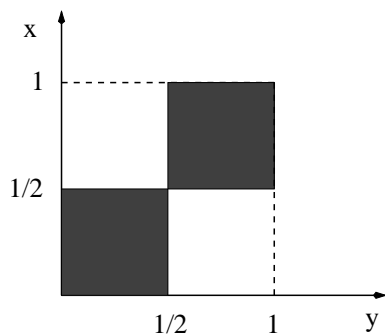
Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region.



- Make fully labeled sketches of the densities $f_X(x)$ and $f_Y(y)$.
- Are X and Y statistically independent? Explain.
- Determine $\hat{X}_{MMSE}(y)$, the minimum mean square error estimator for X , given the observation $Y = y$.

Problem 7.6

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region.



- Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X ?
- Are X and Y uncorrelated? Are X and Y statistically independent?
- Determine $\hat{X}_{MMSE}(y)$, the minimum mean square error estimator for X , given the observation $Y = y$.
- Determine $\hat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X , given the observation $Y = y$.