

Problem Set 8

Jointly Gaussian R.V.'s, Random Processes, Autocorrelation, Stationarity

Issued: Thursday, October 26th. **Due:** Thursday, November 2nd (beginning of lecture).

Reading from Lathi: Chapter 11, Sections 11.1–11.3.

Reading from Haykin: Chapter 4, Sections 4.6–4.12.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 9th, from 1:30pm to 2:50pm in 369 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 2nd. The corresponding material includes Problem Sets 5 through 9 and

(i) **Lathi:** Chapters 5, 10, and 11 (excluding Sections 11.5–11.6), OR

(ii) **Haykin:** Chapter 3 and Chapter 4 (excluding Section 4.13).

For the exam, you can bring *two* 8.5 × 11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 8.1 (Optional)

- (a) Let $X[n]$ be a discrete-time random process defined for $n = 1, 2, 3, \dots$. Each $X[n]$ is independent and has $\Pr(X[n] = 0) = \Pr(X[n] = 1) = 1/2$. Define the random process $Y[n]$ to be

$$Y[n] = X[n] - X[n - 1].$$

Find $E[Y[n]]$ and $\text{Var}[Y[n]]$.

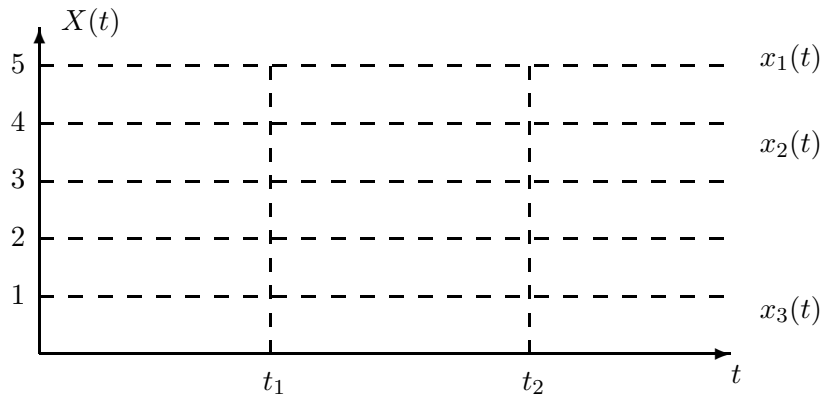
- (b) Let $X(t)$ be a random process defined by $X(t) = At + B$.

- (i) If B is constant and A is a random variable that is uniformly distributed in $(-1, 1)$, sketch a sample function of this process and find $E[X(t)]$.
- (ii) If A is constant and B is a random variable that is uniformly distributed in $(0, 2)$, sketch a sample function of this process and find $E[X(t)]$.
- (iii) If A, B are independent, identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2 , find the joint pdf $f_{X(t_1), X(t_2)}(x_1, x_2)$.

Problem 8.2

A random process $X(t)$ has a sample space with three possible time waveforms as shown below. The probabilities for these three time waveforms are

$$\Pr(x_1(t)) = 1/4, \quad \Pr(x_2(t)) = 1/8, \quad \Pr(x_3(t)) = 5/8.$$



- Determine the pmf $\Pr[X(t_1) = x]$ for the random variable $X(t_1)$.
- Determine the joint pmf $\Pr[X(t_1) = x_1, X(t_2) = x_2]$ for random variables $X(t_1)$ and $X(t_2)$.
- Determine the autocorrelation $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$.

Problem 8.3 (Optional)

A random process $X(t)$ is given by

$$X(t) = A \cos(\omega_0 t + \Theta),$$

where ω_0 is a constant and Θ is random variable that is uniformly distributed between 0 and $\pi/2$. Find $E[X(t)]$.

Problem 8.4

Consider a random process $X(t) = A \cos(\omega_0 t)$ where ω_0 is a constant and A is a random variable that is uniformly distributed in $[0, 1]$.

- Find the autocorrelation $R_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $R_{XX}(t_1, t_2)$ is defined as

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)].$$

- Find the autocovariance $C_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $C_{XX}(t_1, t_2)$ is defined as

$$C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))].$$

- Is $X(t)$ wide-sense stationary?

Problem 8.5

Show that the random process $X(t)$ defined as

$$X(t) = \sin(\Omega t) ,$$

where Ω is a random variable uniformly distributed in the interval $[0, 2\pi W]$ is non-stationary.

Problem 8.6

Let X and Y be identical, statistically independent (i.i.d.) Gaussian random variables with zero mean and unit variance. Define the Gaussian random process $Z(t)$ as

$$Z(t) = X \cos(2\pi t) + Y \sin(2\pi t) .$$

Determine the joint probability density function $f_{Z(t_1), Z(t_2)}(z_1, z_2)$ of random variables $Z(t_1)$ and $Z(t_2)$ (obtained by observing the random process $Z(t)$ at times t_1 and t_2). Is $Z(t)$ stationary?

Problem 8.7

A communication channel has an input signal $S(t)$ which can be modeled as a modulated sine wave with random phase and random amplitude at any given time, i.e.,

$$S(t) = X(t) \sin(\omega_0 t + \Theta) ,$$

where ω_0 is a constant, Θ is a random variable that is uniformly distributed in $[0, 2\pi]$ and is independent of the amplitude, and amplitude $X(t)$ is a wide-sense stationary random process with

$$\mu_X(t) = 0 , \quad -\infty < t < +\infty ,$$

$$R_{XX}(t + \tau, t) \equiv R_{XX}(\tau) = Ae^{-|\tau|} , \quad -\infty < \tau < +\infty .$$

Find the autocorrelation function $R_{SS}(t_1, t_2)$ for the signal $S(t)$. Is $S(t)$ a wide-sense stationary random process?

Problem 8.8

Let $X(t)$ be a wide-sense stationary random process with

$$R_{XX}(\tau) = 2e^{-|\tau|} , \quad -\infty < \tau < +\infty .$$

- (a) What is the average power in the random process $X(t)$?
- (b) Find the value of $E[(X(t+1) - X(t-1))^2]$.
- (c) Let $Y(t)$ be a random process defined by

$$Y(t) = 5X(2t) - X(t-1) , \quad -\infty < t < +\infty .$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ a wide-sense stationary random process?