University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: Communications I

Fall 2001

Problem Set 2

Fourier Transform, Hilbert Transform, Bandpass Systems

Issued: Thursday, Sept. 6th. **Due:** Thursday, Sept. 13th (beginning of lecture).

Reading from Haykin: Appendix 2 (in particular, Sections A2.3–A2.4).

Problem 2.1

(a) Show that, if signal $g(t)$ has Fourier transform $G(f)$, then the Fourier transform of

$$g'(t) = g(t+T) + g(t-T)$$

is given by

$$G'(f) = 2G(f)\cos(2\pi fT) \ .$$

(b) What is the Fourier transform $G(f)$ of the signal

$$g(t) = \text{rect}(t-4) + \text{rect}(t+4) \ ?$$

(c) What is the signal $g(t)$ that has Fourier transform

$$G(f) = \text{rect}(f-4) + \text{rect}(f+4) \ ?$$

Problem 2.2

Consider the following scenario:

- Signal $g_1(t) = 10^3\text{rect}(10^4 t)$ is applied as input to an ideal low-pass filter with frequency response $H_1(f) = \text{rect}(f/20000)$ to produce output $y_1(t)$.
- Signal $g_2(t) = \delta(t)$ is applied as input to an ideal low-pass filter with frequency response $H_2(f) = \text{rect}(f/10000)$ to produce output $y_2(t)$.
- Outputs $y_1(t)$ and $y_2(t)$ are then multiplied to obtain the final output $y(t) = y_1(t)y_2(t)$.

(a) Find $G_1(f)$ and $G_2(f)$.

(b) Find $h_1(t)$ and $h_2(t)$.

(c) Find $Y_1(f)$ and $Y_2(f)$; also, find the bandwidths of $y_1(t)$, $y_2(t)$ and $y(t)$.

Problem 2.3

Show that $\int_{-\infty}^{+\infty} \text{sinc}^2(kx)dx = 1/k$.

Problem 2.4

(a) Show that the complex envelope of the sum of two narrow-band signals (with the same carrier frequency) is equal to the sum of their individual complex envelopes.

(b) Consider a signal of the form

$$s(t) = c(t)m(t) \ ,$$

where $m(t)$ is a low-pass signal whose Fourier transform $M(f)$ is zero for $|f| > W$, and $c(t)$ is a high-pass signal whose Fourier transform $C(f)$ is zero for $|f| < W$. Show that the Hilbert transform of $s(t)$ is given by

$$\hat{s}(t) = \hat{c}(t)m(t) \ ,$$

where $\hat{c}(t)$ is the Hilbert transform of $c(t)$.

Problem 2.5

Let $m(t) = \text{sinc}(t)$ and let $\widehat{m}(t)$ denote its Hilbert transform. Define

$$x(t) = m(t)\cos 2\pi f_c t - \widehat{m}(t)\sin 2\pi f_c t$$

to be a bandpass signal ($f_c >> \frac{1}{2}$).

(a) Find the pre-envelope $x_+(t)$ and the complex envelope $\widetilde{x}(t)$ of signal $x(t)$.

(b) Determine the Fourier transform and bandwidth of $x(t)$.

Problem 2.6

Consider the following *demodulation* process. A low-pass signal $g(t)$ with bandwidth W is recovered from the modulated signal $s(t) = g(t)\cos(2\pi f_c t)$ (with *carrier frequency* f_c) via the following process:

- Multiplication of $s(t)$ by $2\cos(2\pi f_c t)$ to obtain $r(t)$.
- Low-pass filtering of $r(t)$ with an ideal low-pass filter of bandwidth W.

Show that this demodulation process will be successful in recovering $g(t)$ if $W < f_c$.