University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 1999

Mid-Semester Exam II

Thursday, November 11, 1:30–2:50pm, 151 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!!!

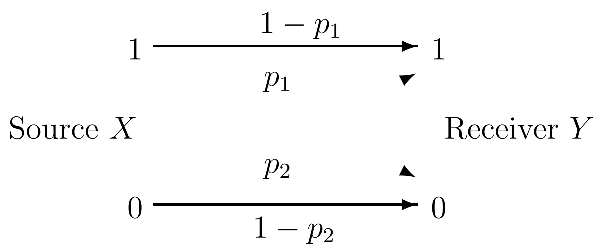
- This is a **closed-book** exam, but **two** sheets of notes (both sides) are allowed. Calculators should not be necessary but feel free to use one.
- **Write your name on the answer booklet.**
- There are **five** problems (which are not equally weighted). The weighting is indicated within each problem.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may lose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Useful Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$
- $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$
- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j2\pi f}, \quad \alpha > 0$
- $\mathcal{FT}\{\text{sinc}(2Wt)\} = \frac{1}{2W}\text{rect}(\frac{f}{2W})$
- $\mathcal{HT}\{\cos(2\pi f_0 t)\} = \sin(2\pi f_0 t)$

Problem 1 (10/80, equally weighted parts)

The discrete, binary and memoryless communication channel shown below can transmit one bit at each time step ("0" or "1"). The number next to an arrow denotes the probability of receiving the bit at the right of the arrow given transmission of the bit at the left of the arrow. The constants p_1 and p_2 satisfy $0 \leq p_1 < \frac{1}{2}$ and $0 \leq p_2 < \frac{1}{2}$.



We use the above channel to send two different symbols (A and B). For each symbol transmission we take two time steps and use two bits: for symbol A we use "00" whereas for symbol B we use "11". The joint probabilities of the transmitted and received sequences are given in the following table (the columns of the table correspond to the two possible transmitted sequences and the rows of the table correspond to the four possible received sequences; each entry gives the joint probability that the sequence specified by the column was transmitted and the sequence specified by the row was received).

Received sequence	Transmitted Sequence A: 00	B: 11
$R_0 = 00$	$\frac{16}{75}$	$\frac{2}{27}$
$R_1 = 01$	$\frac{4}{75}$	$\frac{4}{27}$
$R_2 = 10$	$\frac{4}{75}$	$\frac{4}{27}$
$R_3 = 11$	$\frac{1}{75}$	$\frac{8}{27}$

(a) Based on the received signal R_i, $0 \leq i \leq 3$ (i.e., for each of the four possible two-bit sequences) decide what symbol (A or B) was transmitted in a way that minimizes the overall probability of error. What is the corresponding probability of error?

(b) For this channel, what is p_1, the probability that a transmitted bit "1" is received as a "0"?

Problem 2 (10/80, equally weighted parts)

Consider a system for determining whether a certain communication channel is being used or not. Let H_1 denote the hypothesis that the channel is being used, and let H_0 denote the hypothesis that the channel is not being used. The decision between hypotheses H_1 and H_0 will be based on a single scalar measurement R (e.g., the output of our antenna at a particular time). This measurement R is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 \quad : \quad f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 \quad : \quad f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} \ .$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

(a) Find the decision rule that minimizes the probability of error.

(b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 \ ,$$

where α_1, α_2, α_3, γ_1 and γ_2 are appropriate constants, and $Q(\gamma)$ is defined as

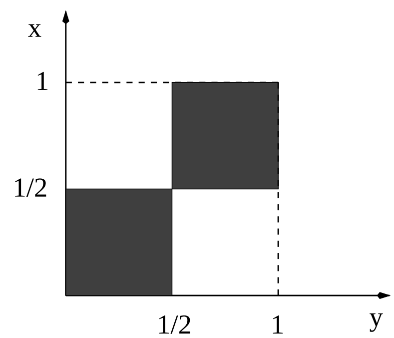
$$Q(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau \ .$$

Note: If you do not have the answer to part (a) you can still receive partial credit if you find the probability of error for the decision rule

$$R \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} 2 \ .$$

Problem 3 (20/80, equally weighted parts)

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region of the figure below.



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Are X and Y uncorrelated? Are X and Y statistically independent?

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

(d) Determine $\widehat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X, given the observation $Y = y$.

Problem 4 (20/80, equally weighted parts)

Let $X[n]$ be a discrete-time random process defined for $n = ..., -1, 0, 1, 2, ...$. Each $X[i]$ is independent from all other samples $X[j]$, $j \neq i$, and has pdf $f_{X[i]}(x)$ that is Gaussian with mean $\mu_X = 0$ and variance $\sigma_X^2 = 1$. Let $Y[n]$ be a random process defined via

$$Y[n] = \frac{4}{5}X[n] + \frac{3}{5}X[n-1] \ .$$

(a) Find $E[Y[n]]$ and $R_{YY}[n_1, n_2]$. Is $Y[n]$ a wide-sense stationary random process?

(b) Find the pdf $f_{W_1}(w_1)$ of random variable $W_1 = Y[6]$.

(c) Find the joint pdf $f_{W_1,W_2}(w_1, w_2)$ of random variables $W_1 = Y[6]$ and $W_2 = Y[8]$.

(d) Find the MMSE of $W_3 = Y[7]$ given $W_1 = Y[6] = y$ (i.e., the one-step linear predictor).

Problem 5 (20/80, equally weighted parts)

Suppose that random process $X(t)$ is given by

$$X(t) = W(t) + V(t) \ ,$$

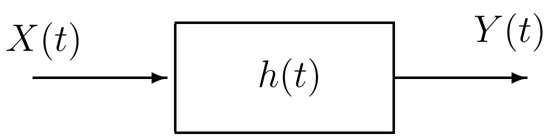
where $W(t)$ and $V(t)$ are zero-mean jointly wide-sense stationary random processes such that

$$S_{VV}(f) = 1 \ , \quad S_{WW}(f) = \frac{16}{9 + (2\pi f)^2} \ , \quad \text{and} \quad R_{VW}(\tau) = 0 \text{ for all } \tau \ .$$

(a) Explain why $X(t)$ is a wide-sense stationary random process.

(b) Find $S_{XX}(f)$ and the corresponding $R_{XX}(\tau)$ (i.e., find the power spectral density of $X(t)$ and the corresponding autocorrelation function).

(c) Suppose $X(t)$ is processed by a stable linear time-invariant system with impulse response $h(t) = e^{-5t}u(t)$. Find the autocorrelation function $R_{YY}(\tau)$ of the output $Y(t)$ of the system.



(d) Find a whitening filter for the random process $X(t)$, i.e., find the impulse response $h_w(t)$ of a stable, linear time-invariant system such that, when input is $X(t)$, the output of the filter is a process $W(t)$ whose autocorrelation function is an impulse.