University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2001

Final Exam

Friday, December 14th, 7:00–10:00pm, 161 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!!!

- This is a **closed-book** exam, but **three** sheets of notes (both sides) are allowed. Calculators should not be necessary but feel free to use one.
- **Write your name on the answer booklet.**
- There are **six** problems (which are not equally weighted).
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may lose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

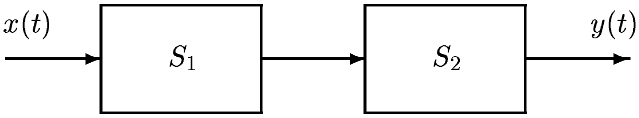
Useful Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$
- $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$
- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j2\pi f}, \quad \alpha > 0,$ $\qquad \mathcal{FT}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - j2\pi f}, \quad \alpha > 0$
- $\mathcal{FT}\{\text{sinc}(2Wt)\} = \frac{1}{2W}\text{rect}(\frac{f}{2W})$

Problem 1 (10/100, equally weighted parts)

This problem has two *independent* parts.

A. Consider the following cascade of two systems S_1 and S_2.



Answer TRUE or FALSE: If S_1 and S_2 are both time-invariant, then the overall system (with input $x(t)$ and output $y(t)$) is necessarily time-invariant.

Explain your reasoning!

B. Let X_1 and X_2 be two random variables that depend on another random variable Y; let $\widehat{Y}(x_1) \equiv \alpha x_1 + \beta$ be the linear mean square error estimator of Y based on the measurement $X_1 = x_1$ (where α and β are appropriate constants); let $\widehat{Y}'(x_1, x_2) \equiv \alpha' x_1 + \beta' x_2 + \gamma'$ be the linear mean square error estimator of Y based on the measurements $X_1 = x_1$ and $X_2 = x_2$ (where α' and β' and γ' are appropriate constants).

Answer TRUE or FALSE: The mean square error $E[(\widehat{Y}(x_1) - Y)^2]$ resulting from the use of estimator $\widehat{Y}(x_1)$ is smaller than the mean square error $E[(\widehat{Y}'(x_1, x_2) - Y)^2]$ resulting from the use of estimator $\widehat{Y}'(x_1, x_2)$.

Explain your reasoning!

Problem 2 (15/100, equally weighted parts)

Consider a wide-sense stationary (WSS) Gaussian random process $N(t)$ with zero mean and power spectral density

$$S_{NN}(f) = \begin{cases} 5, & -\frac{1}{5} \leq f \leq \frac{1}{5}\,, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the autocorrelation function $R_{NN}(\tau)$ of the random process $N(t)$? What is its average power?

(b) What is the pdf $f_Y(y)$ of the random variable $Y = N(2)$, the sample of the random process $N(t)$ at time $t = 2$?

(c) Let $X = N(2)$ and $Y = N(7)$; find the joint density $f_{X,Y}(x, y)$ of random variables X and Y.

Problem 3 (20/100, equally weighted parts)

In this problem we are interested in comparing the performance of a DSB-SC modulation scheme with the performance of an FM modulation scheme. In both cases, the message signal is $m(t) = A_m \cos 2\pi f_m t$ and the modulated signal ($s_{DSB}(t)$ or $s_{FM}(t)$) is corrupted by additive white Gaussian noise so that the received signal $r(t)$ is given by

$$r(t) = s_{DSB}(t) + n(t) \quad \text{or} \quad r(t) = s_{FM}(t) + n(t)\ .$$

The noise $n(t)$ is a sample path from a white Gaussian random process $N(t)$ with zero mean and power spectral density $S_{NN}(f) = \frac{N_0}{2}$. The following parameters are given:

$$A_m = 1 \text{ Volt}, \quad f_m = 10 \text{ KHz}, \quad N_0 = 10^{-5} \text{ W/Hz}.$$

Note: For the purposes of this problem, you can take the signal to noise ratio at the output $y(t)$ of the demodulator to be

$$\text{SNR} = \frac{\text{Power of component of } y(t) \text{ that is due to } m(t)}{\text{Average power of component of } y(t) \text{ that is due to } n(t)}\ .$$

(a) The message signal $m(t)$ is DSB-SC modulated so that the transmitted signal $s_{DSB}(t)$ is given by

$$s_{DSB}(t) = A_d\, m(t) \cos 2\pi f_c t$$

for $f_c = 50$ MHz and $A_d = 10$. The received signal $r(t)$ is demodulated using a coherent demodulator as discussed in class. What is the signal to noise ratio at the output of the coherent demodulator?

(b) The message signal $m(t)$ is frequency modulated so that the transmitted signal $s_{FM}(t)$ is given by

$$s_{FM}(t) = A_f \cos \left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau \right]$$

with the following parameters:

$$f_c = 50 \text{ MHz}, \quad k_f = 30 \text{ KHz/Volt}.$$

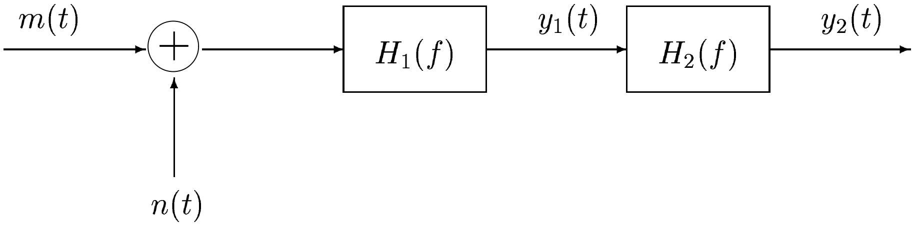
(i) Use Carson's rule to estimate the bandwidth required by $s_{FM}(t)$.

(ii) Let $A_f = 5$ Volt. If the received signal $r(t) = s_{FM}(t) + n(t)$ is demodulated using a standard FM demodulation scheme like the one we studied in class, what is the signal to noise ratio at the output of the demodulator?

(iii) Choose A_f so that $s_{FM}(t)$ has the same power as $s_{DSB}(t)$ in part (a). Find the minimum required bandwidth for $s_{FM}(t)$ so that the signal to noise ratio at the output of the FM demodulator is 6 times the signal to noise ratio at the output of the coherent DSB demodulator.

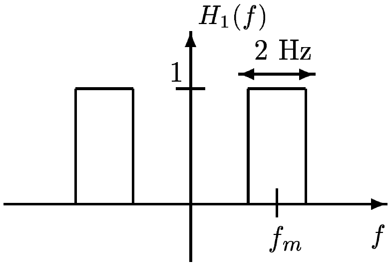
Problem 4 (20/100, equally weighted parts)

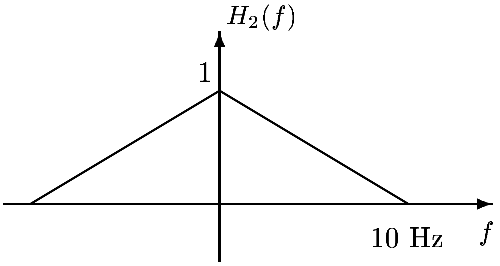
The message signal $m(t) = \cos(2\pi f_m t)$ gets corrupted by additive white Gaussian noise $N(t)$ with zero mean and power spectral density $S_{NN}(f) = 10^{-1}$ W/Hz. The resulting signal gets filtered by the cascade of the two filters shown below.



The frequency response of the filters is given by

$$H_1(f) = \begin{cases} 1, & f_m - 1 \text{ Hz} < |f| < f_m + 1 \text{ Hz} , \\ 0, & \text{otherwise.} \end{cases} \qquad H_2(f) = \begin{cases} 1 - \frac{|f|}{10}, & |f| < 10 \text{ Hz} , \\ 0, & \text{otherwise.} \end{cases}$$





(a) For this part, assume that $f_m = 5$ Hz.

  (i) Find the signal to noise ratio at the output $y_1(t)$ of filter $H_1(f)$.

  (ii) Find the signal to noise ratio at the output $y_2(t)$ of filter $H_2(f)$.

(b) Repeat part (a) for $f_m = 9$ Hz.

Note: The signal to noise ratio is defined as

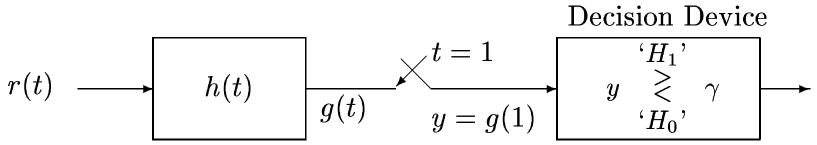
$$\text{SNR} = \frac{\text{Power of component of } y(t) \text{ that is due to } m(t)}{\text{Average power of component of } y(t) \text{ that is due to } n(t)} .$$

Problem 5 (15/100, equally weighted parts)

In this problem we analyze a communication scenario that is based on a variation of the "on-off" signaling scheme that we discussed in class. The transmitter transmits one bit ("0" or "1") by setting the transmitted signal $s(t)$ either to a pulse $p_0(t)$ or to a pulse $p_1(t)$. The signal $r(t)$ received at the receiver is corrupted by additive noise, i.e., $r(t) = s(t) + n(t)$, where $n(t)$ is a sample path of a white Gaussian random process $N(t)$ with zero mean and power spectral density $S_{NN}(f) = \frac{N_0}{2}$.
The receiver is faced with the following binary hypothesis testing problem:

$$\begin{aligned} &\text{Hypothesis } H_0 \text{ ("0" being transmitted)} &:\quad& r(t) = p_0(t) + n(t) \ , \\ &\text{Hypothesis } H_1 \text{ ("1" being transmitted)} &:\quad& r(t) = p_1(t) + n(t) \ . \end{aligned}$$

The prior probabilities for the two hypotheses are equal. In order to make a decision as to whether hypothesis H_0 or H_1 took place, the receiver uses the system shown below.



Threshold γ can be varied (and is to be determined) whereas the impulse response of the linear time-invariant filter is given by

$$h(t) = \begin{cases} 1, & 0 \leq t \leq 1 \ , \\ 0, & \text{otherwise.} \end{cases}$$

We will analyze the performance of this detection scheme for the following two pulses (assume throughout the problem that $A_1 > A_0$):

$$p_0(t) = \begin{cases} A_0 t, & 0 \leq t \leq \frac{1}{2} \ , \\ A_0(1-t), & \frac{1}{2} \leq t \leq 1 \ , \\ 0, & \text{otherwise.} \end{cases} \qquad\qquad p_1(t) = \begin{cases} A_1, & 0 \leq t \leq 1 \ , \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $f_{Y|H_0}(y|H_0)$ and $f_{Y|H_1}(y|H_1)$, the conditional probability densities of Y under hypotheses H_0 and H_1.

(b) Choose γ so that you minimize the probability of error. Express the corresponding probability of error in terms of A_0, A_1 and N_0 and the Q-function.

(c) Find the energies of the pulses $p_0(t)$ and $p_1(t)$. Choose A_0 and A_1 so that you minimize the probability of error while keeping the sum of the energies of $p_0(t)$ and $p_1(t)$ below 12 Joules. What is the corresponding probability of error?

Problem 6 (20/100, equally weighted parts)

In a certain communication system, the transmitted value X is attenuated by a random attenuation and is corrupted by channel noise so that the available measurement Y at the receiving end is related to X as

$$Y = WX \, .$$

The transmitted value X is a uniform random variable in the interval $[-1,\ 1]$ and the attenuation W is a uniform random variable in the interval $[\frac{1}{4},\ \frac{3}{4}]$. Furthermore, X and W are independent.

Hint: The following may make your calculation easier: $E[X] = 0$, $E[X^2] = 1/3$, $E[X^3] = 0$, $E[X^4] = 1/5$, $E[W] = 1/2$, $E[W^2] = 13/48$, $E[W^3] = 5/32$, $E[W^4] = 121/1280$.

(a) Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

(b) One potential improvement to the above estimator is the estimator

$$\widehat{X}'(y) = \alpha' y + \beta' y^2 + \gamma'$$

that allows a term of the form y^2 to enter the picture. Assuming the same conditions as in part (a), find the choices of α', β' and γ' that minimize the mean square error $E[(\widehat{X}'(y) - X)^2]$.