University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2002

Problem Set 7

Estimation, Random Processes, Autocorrelation, Stationarity

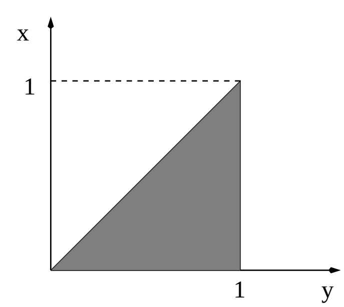
Issued: Thursday, Oct. 24th. **Due:** Thursday, October 31st (beginning of lecture).

Reading from Proakis (2nd Edition): Chapter 4 (up to and including Section 4.2) and lecture notes on MMSE and LMMSE estimation.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 7th, from 1:30pm to 2:50pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, October 31st. The corresponding material will include Problem Sets 1 through 8 and the following reading from Proakis (2nd Edition): Chapters 1, 2, 3 and 4 (up to and including Section 4.4).
During the exam, you can bring two 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.
A copy of an old exam will be available from `http://www.ece.uiuc.edu/ece359` . This sample exam does not necessarily resemble this year's exam (also notice that the material covered in this old exam is slightly different from the material covered in this year's exam).

Problem 7.1 (Optional)

Random variables X and Y have joint pdf $f_{X,Y}(x, y)$ that is constant in the shaded region (and zero elsewhere).



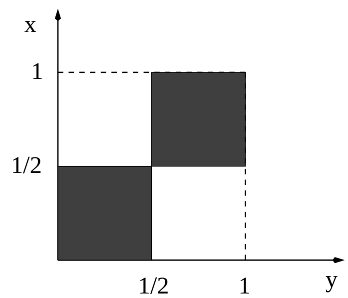
(a) Make fully labeled sketches of the densities $f_X(x)$ and $f_Y(y)$.

(b) Are X and Y statistically independent? Explain.

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

Problem 7.2

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Are X and Y uncorrelated? Are X and Y statistically independent?

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

(d) Determine $\widehat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X, given the observation $Y = y$.

Problem 7.3

In a certain wireless communication system, the transmitted value X is attenuated by a random attenuation and is corrupted by channel noise so that the available measurement Y at the receiving end is related to X as

$$Y = WX + N\ .$$

The transmitted value X is a uniform random variable in the interval $[-1,\ 1]$, the attenuation W is a uniform random variable in the interval $[\frac{1}{2},\ 1]$, and the additive noise N is a Gaussian random variable with zero mean and unit variance. Furthermore, X, W, N are mutually independent.

Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

Hint: The following may make your calculation easier: $E[X] = 0$, $E[X^2] = 1/3$, $E[W] = 3/4$, $E[W^2] = 7/12$.

Problem 7.4

A random process $X(t)$ is given by $X(t) = A\cos(2\pi f_0 t + \Theta)$, where f_0 is a constant and Θ is random variable that is uniformly distributed between 0 and $\pi/2$. Find $E[X(t)]$.

Problem 7.5

(a) Let $X[n]$ be a discrete-time random process defined for $n = 1, 2, 3, \ldots$. Samples $X[1]$, $X[2]$, $X[3]$, ..., are independent, identically distributed (i.i.d.) random variables and have $\Pr(X[n] = 0) = \Pr(X[n] = 1) = 1/2$. Let the random process $Y[n]$ be defined by

$$Y[n] = X[n] - X[n-1] \ .$$

Find $E[Y[n]]$ and $\mathrm{Var}[Y[n]]$.

(b) Let $X(t)$ be a random process defined by $X(t) = At + B$.

(i) If B is constant and A is a random variable that is uniformly distributed in $[-1, 1]$, sketch a sample function of this process and find $E[X(t)]$.

(ii) If A is constant and B is a random variable that is uniformly distributed in $[0, 2]$, sketch a sample function of this process and find $E[X(t)]$.

(iii) If A, B are independent, identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2, find the joint pdf $f_{X(t_1),X(t_2)}(x_1, x_2)$.

Problem 7.6

Consider the random process $X(t) = A\cos(2\pi f_0 t)$, where f_0 is a constant and A is a random variable that is uniformly distributed in $[0, 1]$.

(a) Find the autocorrelation $R_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $R_{XX}(t_1, t_2)$ is defined as $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$.

(b) Find the autocovariance $C_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $C_{XX}(t_1, t_2)$ is defined as $C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$.

(c) Is $X(t)$ wide-sense stationary?

Problem 7.7

Show that the random process $X(t) = \sin(2\pi F t)$, where F is a random variable uniformly distributed in the interval $[0, W]$, is non-stationary.

Problem 7.8

Let $X(t)$ be a wide-sense stationary random process with

$$R_{XX}(\tau) = 2e^{-|\tau|} , \quad -\infty < \tau < +\infty .$$

(a) What is the average power in the random process $X(t)$?

(b) Find the value of $E[(X(t+1) - X(t-1))^2]$.

(c) Let $Y(t)$ be a random process defined by

$$Y(t) = 5X(2t) - X(t-1) , \quad -\infty < t < +\infty .$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ a wide-sense stationary random process?