University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2002

Problem Set 8

WSS Random Processes through LTI Systems, Power Spectral Density

Issued: Thursday, October 31st. **Due:** Never.

Reading: Lecture notes on hypothesis testing; lecture notes on MMSE and LMMSE estimation; Chapter 4 from Proakis (2nd Edition) up to and including Section 4.4.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 7th, from 1:30pm to 2:50pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, October 31st. The corresponding material will include Problem Sets 1 through 8, the lecture notes on hypothesis testing and estimation, and the following reading from Proakis (2nd Edition): Chapters 1, 2, 3 and 4 (up to and including Section 4.4).
During the exam, you can bring two 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.
A copy of an old exam will be available from `http://www.ece.uiuc.edu/ece359` . This sample exam does not necessarily resemble this year's exam (also notice that the material covered in this old exam is slightly different from the material covered in this year's exam).

Problem 8.1

Let $X[n]$ be a discrete-time random process defined for $n = ..., -1, 0, 1, 2, ...$. The samples of the process (i.e., ..., $X[-1]$, $X[0]$, $X[1]$, ...) are independent, identically distributed (i.i.d.) random variables and each $X[i]$ is uniformly distributed in the interval $[-1, 1]$. Define the random process $Y[n]$ to be

$$Y[n] = \frac{2}{3}X[n] + \frac{1}{3}X[n-1] \; .$$

(a) Are $Y[n_1]$ and $Y[n_2]$ for $n_1 \neq n_2$ independent?

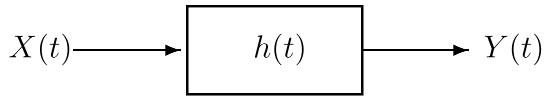
(b) Find $E[Y[n]]$ and $R_{YY}[n_1, n_2]$.

(c) Is $Y[n]$ a wide-sense stationary (WSS) random process?

(d) Find the LMMSE of $Y[5]$ given that $Y[4] = y$.

Problem 8.2

Suppose that a wide-sense stationary (WSS) random process $X(t)$ with zero mean and autocorrelation function $R_{XX}(\tau) = 2e^{-|\tau|}$ is processed as shown below, where the impulse response of the LTI filter is given by $h(t) = e^{-5t}u(t)$.



(a) Find $R_{YY}(\tau)$, the autocorrelation function of $Y(t)$.

(b) What is the average power of the random process $Y(t)$?

Problem 8.3

A communication channel has an input signal $S(t)$ that can be modeled as a modulated sinusoidal wave with random phase and random, time-varying amplitude at any given time, i.e.,

$$S(t) = X(t)\sin(2\pi f_c t + \Theta) \ ,$$

where f_c is a constant, Θ is a random variable that is uniformly distributed in $[0, 2\pi]$, and $X(t)$ is a wide-sense stationary (WSS) random process that is independent from the phase and satisfies

$$\mu_X(t) = 0 \ , \qquad -\infty < t < +\infty \ ,$$

$$R_{XX}(t+\tau, t) \equiv R_{XX}(\tau) = Ae^{-|\tau|} \ , \qquad -\infty < \tau < +\infty \ .$$

Find the autocorrelation function $R_{SS}(t_1, t_2)$ for the signal $S(t)$. Is $S(t)$ a wide-sense stationary (WSS) random process?

Problem 8.4

Consider the random process $X(t) = A$, where A is a random variable that is uniformly distributed in the interval $[-1, 1]$.

(a) Determine $\mu_X(t)$ and $R_{XX}(t_1, t_2)$. Is $X(t)$ a wide-sense stationary (WSS) random process?

(b) Is $X(t)$ ergodic? Justify your answer.

Problem 8.5

A zero-mean Gaussian random process $X(t)$ has power spectral density

$$S_{XX}(f) = \frac{4}{1 + (2\pi f)^2} \ , \qquad -\infty < f < +\infty \ .$$

(a) Determine $R_{XX}(\tau)$, the autocorrelation function of the random process $X(t)$.

(b) The random process $X(t)$ is passed through a stable LTI system with frequency response

$$H(f) = \begin{cases} 1\,, & |f| < 1/\pi\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Determine the average power $E[Y^2(t)]$ of the output random process $Y(t)$.

Problem 8.6

(a) Let random processes $X(t)$ and $Y(t)$ be the input and output respectively of a stable LTI system with frequency response $H(f)$. Assume $X(t)$ is wide-sense stationary (WSS) and define random process $Z(t)$ to be $Z(t) = Y(t) - X(t)$. Find $S_{ZZ}(f)$, the power spectral density of $Z(t)$, in terms of $H(f)$ and $S_{XX}(f)$.

(b) The input $X(t)$ to a stable LTI filter with frequency response

$$H(f) = \frac{2}{2 + j2\pi f}$$

can be modeled as a wide-sense stationary (WSS) random process with zero mean and autocorrelation function $R_{XX}(\tau) = 2e^{-|\tau|}$. Find $S_{ZZ}(f)$, the power spectral density of random process $Z(t) = Y(t) - X(t)$, where $Y(t)$ is the output of the filter.

Problem 8.7

A wide-sense stationary (WSS) random process $X(t)$ with autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$ is processed by a stable LTI system with real-valued impulse response $h(t)$.

(a) For this part, assume that the output of the filter $Y(t)$ is a wide-sense stationary (WSS) random process with autocorrelation function $R_{YY}(\tau) = 3e^{-3|\tau|}$.

  (i) Find $|H(f)|$, the magnitude of the frequency response of the filter.

  (ii) Suppose that $h(t)$ is causal. Find a possible impulse response $h(t)$ for the LTI system. Is your answer unique?

(b) Suppose that the LTI system is known to be stable and that

$$R_{YX}(\tau) = e^{-\tau}u(\tau) - 2e^{-2\tau}u(\tau) + e^{-3\tau}u(\tau)\,.$$

Find a possible impulse response $h(t)$. Is your answer unique?