

Problem Set 1

Linear Time-Invariant Systems, Fourier Transform, Hilbert Transform

Issued: Thursday, Sept. 2nd.

Due: Beginning of lecture on Thursday, Sept. 9th.

Reading from Lathi: Chapter 1, Chapter 2 (excluding Sections 2.5–2.7) and Chapter 3 (excluding Sections 3.4–3.6 and 3.8). The material in Chapter 1 is introductory and we will be revisiting it in more detail during the course of the semester. Most of the material in Chapters 2 and 3 should be already familiar to you.

Reading from Haykin (3rd Edition): Chapter 1 and Chapter 2 (Sections 2.1–2.9). Most of the material in Chapter 2 should be familiar to you from previous courses.

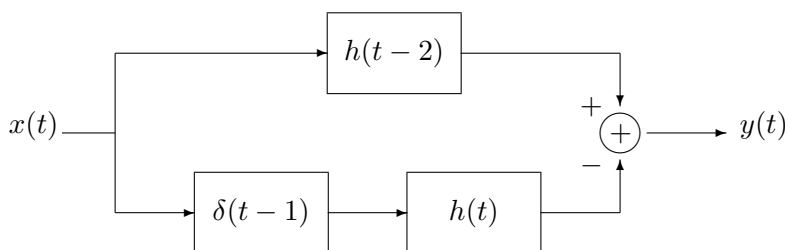
Problem 1.1

- (a) Consider an LTI system with input $x(t)$ and output $y(t)$ related through the equation

$$y(t) = \int_{-\infty}^t e^{-(t-\tau)} x(\tau - 3) d\tau.$$

What is the impulse response $h(t)$ for this system?

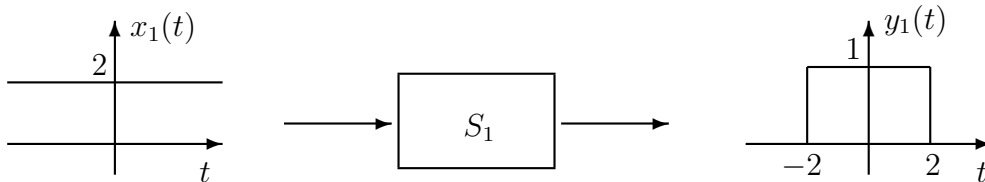
- (b) Determine the response of this system when the input $x(t)$ satisfies $x(t) = 1$ for $-1 \leq t \leq 2$ and $x(t) = 0$ otherwise.
- (c) Consider the following interconnection of LTI systems:



Here $h(t)$ is as in part (a). Determine the output $y(t)$ when the input $x(t)$ is given as in part (b). (Hint: You do *not* need to evaluate a convolution integral.)

Problem 1.2

- (a) The following system S_1 (*not necessarily LTI*) is known to have the input-output pair shown below:



Is system S_1 time-invariant? Explain.

- (b) A *nonlinear* system S_2 is known to be time-invariant. Is the output $y_2(t)$ of system S_2 guaranteed to be periodic in t when input $x_2(t) = \sin(\omega_0 t)$ is applied? Explain.

Problem 1.3

Find the energy and half-power (3dB) bandwidth of the signal $x(t) = e^{-4t}u(t)$.

Problem 1.4

Problem 3.1-1 from Lathi, p. 142.

Problem 1.5

Consider the signal $x(t) = A \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$. Find the Fourier transforms of the even part $x_e(t)$ and the odd part $x_o(t)$ of $x(t)$, defined as

$$\begin{aligned}x_e(t) &= \frac{1}{2} [x(t) + x(-t)] , \\x_o(t) &= \frac{1}{2} [x(t) - x(-t)] .\end{aligned}$$

Problem 1.6

A signal $x(t)$ is applied to a square-law device whose output $y(t)$ is defined by

$$y(t) = x^2(t) .$$

If the spectrum of $x(t)$ is limited to the frequency interval $-W \leq \omega \leq W$, show that the spectrum of $y(t)$ is limited to $-2W \leq \omega \leq 2W$.

Problem 1.7 (Optional)

- (a) Find the Fourier transform of the *Gaussian* signal

$$x(t) = Ae^{-t^2/2T^2}, \quad -\infty < t < +\infty.$$

- (b) If this signal goes through an LTI system with frequency response

$$H(\omega) = e^{-\frac{\omega}{2B^2}}, \quad -\infty < \omega < +\infty,$$

what is the output $y(t)$? Simplify the expression for $y(t)$ for $T \gg 1/B$.

Problem 1.8

- (a) Show that, if signal $g(t)$ has Fourier transform $G(\omega)$, then the Fourier transform of

$$g'(t) = g(t+T) + g(t-T)$$

is given by

$$G'(\omega) = 2G(\omega) \cos(\omega T).$$

- (b) What is the Fourier transform $G(\omega)$ of the signal

$$g(t) = \text{rect}(t-4) + \text{rect}(t+4) ?$$

Note that the function $\text{rect}(t)$ is defined in the textbook as

$$\text{rect}(t) = \begin{cases} 1, & \text{if } |t| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (c) What is the signal $g(t)$ that has Fourier transform

$$G(\omega) = \text{rect}(\omega-4) + \text{rect}(\omega+4) ?$$

Problem 1.9

Consider the following scenario:

- Signal $g_1(t) = 10^3 \text{rect}(10^4 t)$ is applied as input to an ideal low-pass filter with frequency response $H_1(\omega) = \text{rect}(\omega/20000)$ to produce output $y_1(t)$.
- Signal $g_2(t) = \delta(t)$ is applied as input to an ideal low-pass filter with frequency response $H_2(\omega) = \text{rect}(\omega/10000)$ to produce output $y_2(t)$.

- Outputs $y_1(t)$ and $y_2(t)$ are then multiplied to obtain the final output $y(t) = y_1(t)y_2(t)$.
- (a) Find $G_1(\omega)$ and $G_2(\omega)$.
- (b) Find $h_1(t)$ and $h_2(t)$.
- (c) Find $Y_1(\omega)$ and $Y_2(\omega)$; also, find the bandwidths of $y_1(t)$, $y_2(t)$ and $y(t)$.

Problem 1.10

A linear time-invariant system S is known to be stable. Show that, if the input $x(t)$ of S has finite energy, then the output $y(t)$ also has finite energy. In other words, show that if

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty ,$$

then

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt < \infty .$$

Problem 1.11 (Optional)

Show that $\int_{-\infty}^{+\infty} \text{sinc}^2(kx) dx = \pi/k$.

Problem 1.12

- (a) Show that the Hilbert transform of an odd signal is an even function and that the Hilbert transform of an even signal is odd.
- (b) Show that the energy content of a signal is equal to the energy content of its Hilbert transform.