University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: COMMUNICATIONS I

Fall 2004

Problem Set 6

Linear MMSE Estimation, Jointly Gaussian R.V.'s, Random Processes, Autocorrelation, Stationarity

Issued: Thursday, Oct. 21st. **Due:** Thursday, Oct. 28th (beginning of lecture).

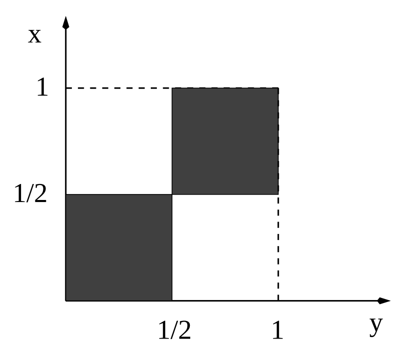
Reading from Lathi: Chapter 10 and Chapter 11, Sections 11.1–11.3.

Reading from Haykin (3rd Edition): Chapter 4, Sections 4.6–4.12.

Announcement: The second Mid-Semester Exam will be held on Tuesday, November 9th, from 5:00pm to 7:00pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 4th. The corresponding material includes Problem Sets 4 through 7 and
(i) **Lathi**: Chapters 5, 10, and 11 (excluding Section 11.6), OR
(ii) **Haykin (3rd Edition)**: Chapters 3 and 4 (excluding Section 4.15).
For the exam, you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 6.1

Random variables X and Y have joint pdf $f_{X,Y}(x, y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Are X and Y uncorrelated? Are X and Y statistically independent?

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

(d) Determine $\widehat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X, given the observation $Y = y$.

Problem 6.2

In a certain wireless communication system, the transmitted value X is attenuated by a random attenuation and is corrupted by channel noise so that the available measurement Y at the receiving end is related to X as

$$Y = WX + N\ .$$

The transmitted value X is a uniform random variable in the interval $[-1,\ 1]$, the attenuation W is a uniform random variable in the interval $[\frac{1}{2},\ 1]$, and the additive noise N is a Gaussian random variable with zero mean and unit variance. Furthermore, X, W, N are mutually independent.

Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

Hint: The following may make your calculation easier: $E[X] = 0$, $E[X^2] = 1/3$, $E[W] = 3/4$, $E[W^2] = 7/12$.

Problem 6.3 (Optional)

(a) Let $X[n]$ be a discrete-time random process defined for $n = 1, 2, 3, \ldots$. Samples $X[n]$ are independent, identically distributed (i.i.d.) random variables and satisfy $\Pr(X[n] = 0) = \Pr(X[n] = 1) = 1/2$. Define the random process $Y[n]$ to be

$$Y[n] = X[n] - X[n-1]\ .$$

Find $E[Y[n]]$ and $\mathrm{Var}[Y[n]]$.

(b) Let $X(t)$ be a random process defined by $X(t) = At + B$.

(i) If B is constant and A is a random variable that is uniformly distributed in $(-1, 1)$, sketch a sample function of this process and find $E[X(t)]$.

(ii) If A is constant and B is a random variable that is uniformly distributed in $(0, 2)$, sketch a sample function of this process and find $E[X(t)]$.

(iii) If A, B are independent, identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2, find the joint pdf $f_{X(t_1),X(t_2)}(x_1, x_2)$.

Problem 6.4

A random process $X(t)$ is given by

$$X(t) = A\cos(\omega_0 t + \Theta) \ ,$$

where ω_0 is a constant and Θ is random variable that is uniformly distributed between 0 and $\pi/2$. Find $E[X(t)]$.

Problem 6.5

Consider a random process $X(t) = A\cos(\omega_0 t)$ where ω_0 is a constant and A is a random variable that is uniformly distributed in $[0, 1]$.

(a) Find the autocorrelation $R_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $R_{XX}(t_1, t_2)$ is defined as

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] \ .$$

(b) Find the autocovariance $C_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $C_{XX}(t_1, t_2)$ is defined as

$$C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] \ .$$

(c) Is $X(t)$ wide-sense stationary?

Problem 6.6

Show that the random process $X(t)$ defined as

$$X(t) = \sin(\Omega t) \ ,$$

where Ω is a random variable uniformly distributed in the interval $[0, 2\pi W]$ is non-stationary.

Problem 6.7

Problem 11.1-7 from Lathi, p. 526.

Problem 6.8

Let X and Y be independent, identically distributed (i.i.d.) Gaussian random variables with zero mean and unit variance. Define the Gaussian random process $Z(t)$ as

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t) \ .$$

Determine the joined probability density function $f_{Z(t_1),Z(t_2)}(z_1, z_2)$ of random variables $Z(t_1)$ and $Z(t_2)$ (obtained by observing the random process $Z(t)$ at times t_1 and t_2). Is $Z(t)$ stationary?

Problem 6.9

A communication channel has an input signal $S(t)$ which can be modeled as a modulated sine wave with random phase and random amplitude at any given time, i.e.,

$$S(t) = X(t)\sin(\omega_0 t + \Theta) \ ,$$

where ω_0 is a constant, Θ is a random variable that is uniformly distributed in $[0, 2\pi]$ and is independent of the amplitude, and amplitude $X(t)$ is a wide-sense stationary random process with

$$\mu_X(t) = 0 \ , \qquad -\infty < t < +\infty \ ,$$

$$R_{XX}(t+\tau, t) \equiv R_{XX}(\tau) = Ae^{-|\tau|} \ , \qquad -\infty < \tau < +\infty \ .$$

Find the autocorrelation function $R_{SS}(t_1, t_2)$ for the signal $S(t)$. Is $S(t)$ a wide-sense stationary random process?

Problem 6.10

Let $X(t)$ be a wide-sense stationary random process with

$$R_{XX}(\tau) = 2e^{-|\tau|} \ , \quad -\infty < \tau < +\infty \ .$$

(a) What is the average power in the random process $X(t)$?

(b) Find the value of $E[(X(t+1) - X(t-1))^2]$.

(c) Let $Y(t)$ be a random process defined by

$$Y(t) = 5X(2t) - X(t-1) \ , \quad -\infty < t < +\infty \ .$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ a wide-sense stationary random process?