University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: COMMUNICATIONS I

Fall 2004

Problem Set 9

Signal to Noise Ratio, Signal Detection in Noise, Noise Analysis in Analog Modulation Schemes

Issued: Thursday, Nov. 18th. **Due:** Thursday, Dec. 2nd (beginning of lecture).

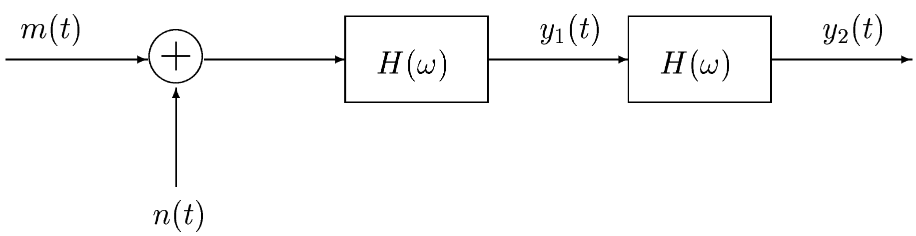
Reading from Lathi: Chapter 13, Sections 13.1–13.2; Chapter 12, Sections 12.1–12.3.

Reading from Haykin (3rd Edition): Chapter 7, Sections 7.1–7.3; Chapter 5, Sections 5.1–5.3 and 5.5–5.7.

Announcement: The Final Exam will be held on Monday, December 13th, from 1:30pm to 4:30pm in 165 Everitt Laboratory. The exam will cover all material from the beginning of the term. For the exam, you can bring *three* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 9.1

The message signal $m(t) = \cos(\omega_m t)$ gets corrupted by additive white Gaussian noise $N(t)$ with zero mean and power spectral density $S_{NN}(\omega) = 10^{-1}$ W/rad/s. The resulting signal gets filtered by the cascade of two identical filters with frequency response $H(\omega)$ as shown below. Assume that $\omega_m = 6$ rad/s.

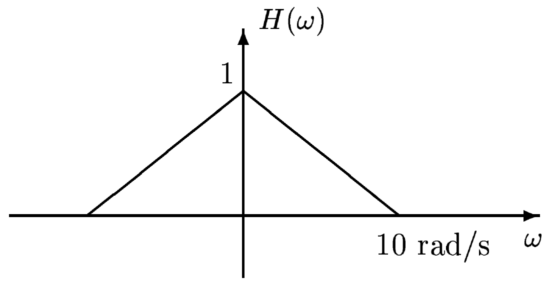


The frequency response of the filter is given by

$$H(\omega) = \begin{cases} 1 - \frac{|\omega|}{10}, & |\omega| < 2\pi \times 10 \text{ rad/s}, \\ 0, & \text{otherwise.} \end{cases}$$

For the purposes of this problem the signal to noise ratio is defined as

$$\text{SNR} = \frac{\text{Power of component of } y_i(t) \text{ that is due to } m(t)}{\text{Average power of component of } y_i(t) \text{ that is due to } n(t)} \; .$$



(a) Find the signal to noise ratio at the output $y_1(t)$.

(b) Find the signal to noise ratio at the output $y_2(t)$.

Problem 9.2

A transmitter transmits one bit ("0" or "1") by setting the transmitted signal $s(t)$ either to zero or the pulse $p(t)$ shown below.

$$p(t) = \begin{cases} 1, & 0 \le t \le 1\,, \\ 0, & \text{otherwise.} \end{cases}$$

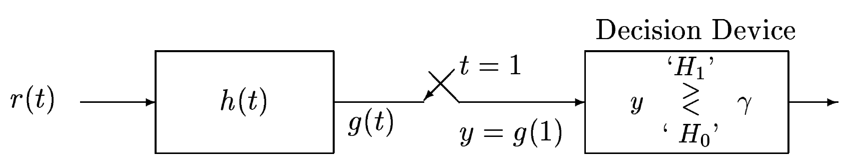
The signal $r(t)$ received at the receiver is corrupted by additive noise, i.e., $r(t) = s(t) + n(t)$, where $n(t)$ is a sample path of a (*non-white*) Gaussian random process $N(t)$ with zero mean and autocorrelation function $R_{NN}(\tau)$ shown below.

$$R_{NN}(\tau) = \begin{cases} 2(2-\tau), & 0 \le \tau \le 2\,, \\ 2(2+\tau), & -2 \le \tau \le 0\,, \\ 0, & \text{otherwise.} \end{cases}$$

The receiver is faced with the following binary hypothesis testing problem:

$$\begin{aligned} &\text{Hypothesis } H_0 \text{ ("0" being transmitted)} &:& \quad r(t) = n(t)\,, \\ &\text{Hypothesis } H_1 \text{ ("1" being transmitted)} &:& \quad r(t) = p(t) + n(t). \end{aligned}$$

The prior probabilities for the two hypotheses are unequal with $\Pr(H_0) = 2\Pr(H_1)$. In order to make a decision as to whether hypothesis H_0 or H_1 took place, the receiver uses the system shown below.



We will analyze the performance of this detection scheme for the following receiver filter $h(t)$:

$$h(t) = \begin{cases} 1, & 0 \le t \le 1\,, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $f_{Y|H_0}(y|H_0)$, the conditional probability density of Y under hypothesis H_0.

(b) Find $f_{Y|H_1}(y|H_1)$, the conditional probability density of Y under hypothesis H_1.

(c) Choose the threshold γ so that the probability of error is minimized.

Problem 9.3

Problem 12.1-1 from Lathi, p. 572.

Problem 9.4

Problem 12.2-1 from Lathi, p. 573.

Problem 9.5

Problem 12.2-4 from Lathi, p. 573.

Problem 9.6

Problem 12.3-1 from Lathi, pp. 573–574.

Problem 9.7

In this problem we are interested in comparing the performance of a DSB-SC modulation scheme with the performance of an FM modulation scheme. In both cases, the message signal is $m(t) = A_m \cos \omega_m t$ and the modulated signal ($s_{DSB}(t)$ or $s_{FM}(t)$) is corrupted by additive white Gaussian noise so that the received signal $r(t)$ is given by

$$r(t) = s_{DSB}(t) + n(t) \quad \text{or} \quad r(t) = s_{FM}(t) + n(t)\ .$$

The noise $n(t)$ is a sample path from a white Gaussian random process $N(t)$ with zero mean and power spectral density $S_{NN}(\omega) = \frac{N_0}{2}$. The following parameters are given:

$$A_m = 1 \text{ Volt}, \quad \omega_m = 2\pi \times 10 \text{ rad/s}, \quad N_0 = 10^{-5} \text{ W/rad/s}.$$

For the purposes of this problem, you can take the signal to noise ratio at the output $y(t)$ of the demodulator to be

$$\text{SNR} = \frac{\text{Power of component of } y(t) \text{ that is due to } m(t)}{\text{Average power of component of } y(t) \text{ that is due to } n(t)}\ .$$

(a) The message signal $m(t)$ is DSB-SC modulated so that the transmitted signal $s_{DSB}(t)$ is given by

$$s_{DSB}(t) = A_d\, m(t) \cos \omega_c t$$

for $\omega_c = 2\pi \times 50$ Mrad/s and $A_d = 10$. The received signal $r(t)$ is demodulated using a coherent demodulator as discussed in class. What is the signal to noise ratio at the output of the coherent demodulator?

(b) The message signal $m(t)$ is frequency modulated so that the transmitted signal $s_{FM}(t)$ is given by

$$s_{FM}(t) = A_f \cos \left[\omega_c t + k_f \int_{-\infty}^{t} m(\tau) d\tau \right]$$

with $\omega_c = 2\pi \times 50$ Mrad/s and $k_f = 2\pi \times 30$rad/s/Volt.

(i) Use Carson's rule to estimate the bandwidth required by $s_{FM}(t)$.

(ii) Let $A_f = 5$ Volt. If the received signal $r(t) = s_{FM}(t) + n(t)$ is demodulated using a standard FM demodulation scheme like the one we studied in class, what is the signal to noise ratio at the output of the demodulator?

(iii) Choose A_f so that $s_{FM}(t)$ has the same power as $s_{DSB}(t)$ in part (a). Find the minimum required bandwidth for $s_{FM}(t)$ so that the signal to noise ratio at the output of the FM demodulator is 6 times the signal to noise ratio at the output of the coherent DSB demodulator.