University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2000

Mid-Semester Exam II

Thursday, November 9, 1:30–2:50pm, 161 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!!!

- This is a **closed-book** exam, but **two** sheets of notes (both sides) are allowed. Calculators should not be necessary but feel free to use one.
- **Write your name on the answer booklet.**
- There are **four** equally weighted problems for a total of **80 points**.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may lose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Useful Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$
- $\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j\omega}, \quad \alpha > 0, \qquad \mathcal{FT}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - j\omega}, \quad \alpha > 0$
- $\mathcal{FT}\{\frac{W}{\pi}\text{sinc}(Wt)\} = \text{rect}(\frac{\omega}{2W})$
- $\mathcal{HT}\{\cos(\omega_0 t)\} = \sin(\omega_0 t)$

Problem 1 (20/80, equally weighted parts)

This problem has two independent parts.

Part A. Consider the FM signal

$$s_{FM}(t) = A_c \cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\tau)d\tau\right] ,$$

where $m(t) = A_m \cos(\omega_m t)$. The following parameters are given:

$A_c = 20V, \quad \omega_c = 2\pi\times 10^8 \text{rad/s}, \quad k_f = 2\pi\times 10^2 \text{rad/sVolt}, \quad A_m = 10V, \quad \omega_m = 2\pi\times 10^3 \text{rad/s}$.

(i) What is the power of $s_{FM}(t)$?

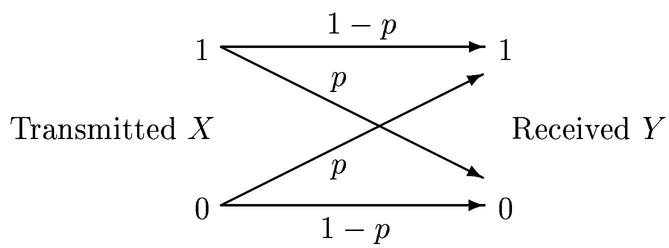
(ii) Find the bandwidth of $s_{FM}(t)$ using Carson's rule.

Part B. Determine whether the following statements are TRUE or FALSE and give a brief explanation.

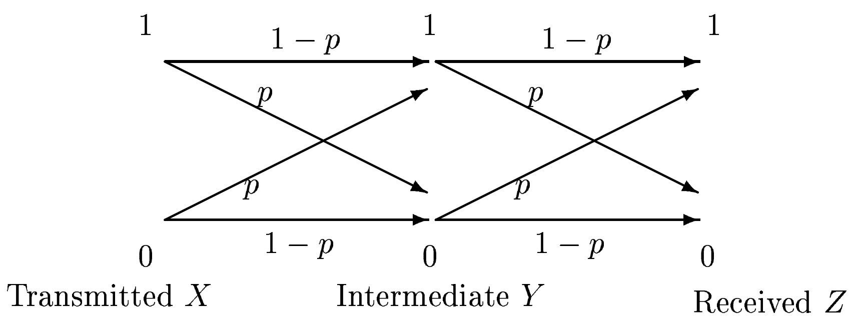
(i) If $E[XY] = E[X]E[Y]$, then random variables X and Y are independent.

(ii) If random variables X and Y are uncorrelated (i.e., if $E[XY] = E[X]E[Y]$), then their sum $Z = X + Y$ is a random variable with variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$, where σ_X^2 is the variance of X and σ_Y^2 is the variance of Y.

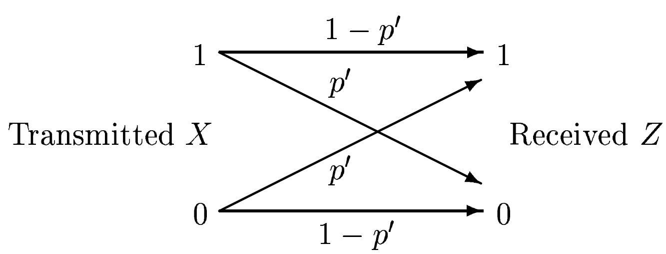
Problem 2 (20/80, equally weighted parts)



The discrete, binary, symmetric and memoryless communication channel shown above can transmit one bit (each time step, X could be 0 or 1). The number next to an arrow denotes the conditional probability of receiving the bit to the right of the arrow given transmission of the bit to the left of the arrow. The constant p is given by $p = \frac{1}{4}$.
Consider a cascade of two such channels connected as shown below.



(a) Show that the cascaded channels can be simplified to a single discrete, binary, symmetric and memoryless channel as shown below, and find the value for p'.



(b) The channel in part (a) is used to transmit one of two 3-bit sequences: sequence $s_0 = 000$ is transmitted with probability $\Pr(s_0) = 4/5$ and sequence $s_1 = 111$ is transmitted with probability $\Pr(s_1) = 1/5$. Suppose that the receiver sees "101" and needs to decide which of the two sequences (s_0 or s_1) was transmitted *in a way that minimizes the probability of error*. What should the receiver decide after observing the sequence "101"?

Note: If you do not have the answer to part (a), use $p' = \frac{1}{4}$ to receive credit for part (b).

Problem 3 (20/80, equally weighted parts)

In a simple binary communication system, during every T seconds, one of two possible signals $s_0(t)$ and $s_1(t)$ is received. Let H_0 denote the hypothesis that $s_0(t)$ is sent and let H_1 denote the hypothesis that $s_1(t)$ is sent. We assume that

$$s_0(t) = +1 \quad \text{and} \quad s_1(t) = -1 \quad \text{for} \quad 0 \le t \le T\,.$$

The decision between hypotheses H_0 and H_1 is based on a single scalar measurement R at time $T/2$. This measurement R is corrupted by noise so that under each of the hypothesis the following is true:

$$H_0 \; : \; f_{R|H_0}(r|H_0) = \frac{1}{2}e^{-|r-1|}\,,$$

$$H_1 \; : \; f_{R|H_1}(r|H_1) = \frac{1}{2}e^{-|r+1|}\,.$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = 2/3$ and $\Pr(H_1) = 1/3$.

(a) Given the observation $R = r$, find the decision rule that minimizes the probability of error. Simplify your decision rule to a rule in the form

$$r \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} \gamma$$

for an appropriate constant γ.

(b) What is the probability of error for the decision rule of part (a)?

Note: If you do not have the answer to part (a) you can still receive partial credit if you find the probability of error for the decision rule

$$r \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} 0\,.$$

Problem 4 (20/80, equally weighted parts)

This problem has two independent parts.

Part A. In a certain communication system, a transmitted value X is attenuated by a random attenuation W so that the output of the channel is given by $Y = X/W$. The transmitted value X is a uniform random variable in the interval $[-1, 1]$ and the attenuation W is a uniform random variable in the interval $[1, 2]$. Assume that X and W are independent.

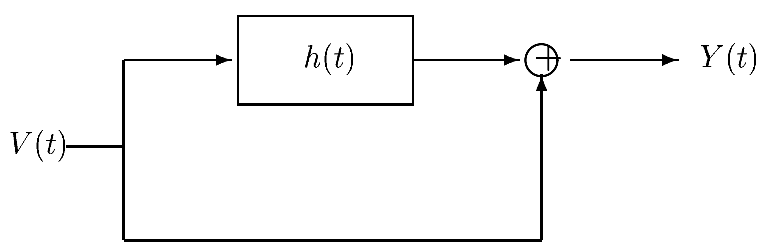
Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

Hint: Some of the following may make your calculation easier: $E[X] = 0$, $E[W] = 3/2$, $E[1/W] = \ln 2$, $E[X^2] = 1/3$, $E[W^2] = 7/3$, $E[1/W^2] = 1/2$.

Part B. Suppose that a wide-sense stationary random process $V(t)$ is processed as shown below, where the impulse response of the LTI filter is given by $h(t) = e^{-2t}u(t)$.



(i) Is $Y(t)$ a wide-sense stationary random process? Explain.

(ii) Assume that $V(t)$ is zero-mean with autocorrelation $R_{VV}(\tau) = e^{-|\tau|}$. Find the average power of $Y(t)$.