University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2005

Problem Set 1

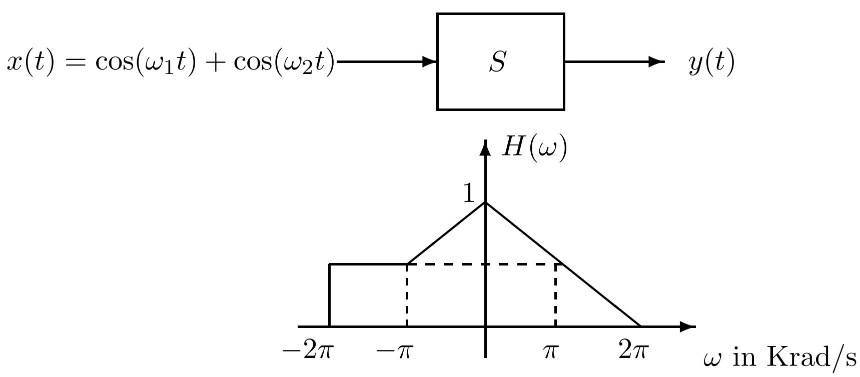
Linear Time-Invariant Systems, Fourier Transform, Hilbert Transform

Issued: Tuesday, Aug. 30th. **Due:** Beginning of lecture on Thursday, Sept. 8th.

Reading from Lathi: Chapter 1, Chapter 2 (excluding Sections 2.5–2.7) and Chapter 3 (excluding Sections 3.4–3.6 and 3.8). The material in Chapter 1 is introductory and we will be revisiting it in more detail during the course of the semester. Most of the material in Chapters 2 and 3 should be already familiar to you.

Problem 1.1

The input signal $x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ with $\omega_1 = 2\pi \times 250$ rad/s and $\omega_2 = 2\pi \times 750$ rad/s is applied to the linear time-invariant (LTI) system S with frequency response $H(\omega)$ as indicated below. Determine $y(t)$.



Problem 1.2

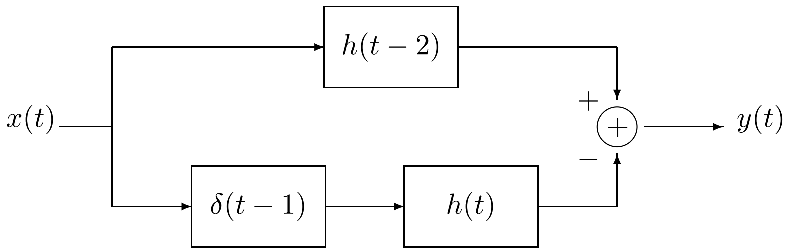
(a) Consider an LTI system with input $x(t)$ and output $y(t)$ related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 3) d\tau.$$

What is the impulse response $h(t)$ for this system?

(b) Determine the response of this system when the input $x(t)$ satisfies $x(t) = 1$ for $-1 \le t \le 2$ and $x(t) = 0$ otherwise.

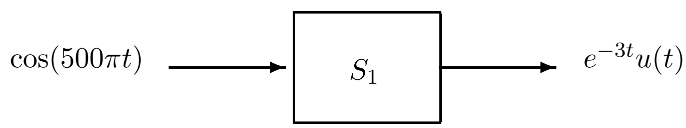
(c) Consider the following interconnection of LTI systems:



Here $h(t)$ is as in part (a). Determine the output $y(t)$ when the input $x(t)$ is given as in part (b). (Hint: You do *not* need to evaluate a convolution integral.)

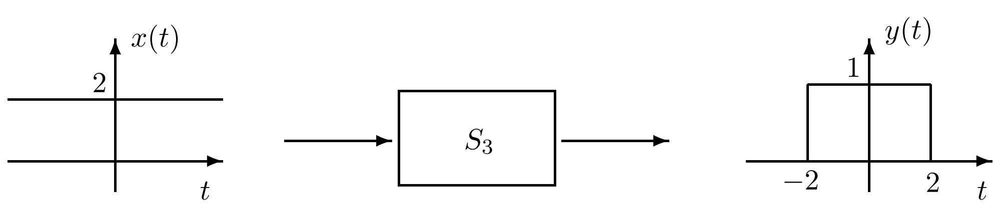
Problem 1.3

(a) System S_1 shown below is associated with the indicated input/output pair. Determine whether the system (i) could not, (ii) could or (iii) must be linear time-invariant (LTI). Choose the strongest statement that applies and *justify* your answer. If you decide that the system could or must be LTI, determine a possible frequency response for it.



(b) A *nonlinear* system S_2 is known to be time-invariant. Is the output $y_2(t)$ of system S_2 guaranteed to be periodic in t when input $x_2(t) = \sin(\omega_0 t)$ is applied? Explain.

(c) The following system S_3 (*not necessarily* LTI) is known to have the input-output pair shown below:



Is system S_3 time-invariant? Explain.

Problem 1.4

(a) Find the half-power bandwidth of the signal $x(t)$ given by

$$x(t) = \delta(t) + e^{-2t}u(t) \ .$$

(b) Find the energy of the signal $x(t)$ given by

$$x(t) = e^{-\alpha t}u(t) - e^{-\beta t}u(t)$$

for $\alpha > \beta > 0$.

Problem 1.5

Problem 3.1–1 from Lathi, p. 142.

Problem 1.6

Consider the signal $x(t) = A \ \mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$. Find the Fourier transforms of the even part $x_e(t)$ and the odd part $x_o(t)$ of $x(t)$, defined as

$$\begin{aligned} x_e(t) &= \frac{1}{2}\left[x(t) + x(-t)\right] \ , \\ x_o(t) &= \frac{1}{2}\left[x(t) - x(-t)\right] \ . \end{aligned}$$

Problem 1.7

A signal $x(t)$ is applied to a square-law device whose output $y(t)$ is defined by

$$y(t) = x^2(t) \ .$$

If the spectrum of $x(t)$ is limited to the frequency interval $-W \leq \omega \leq W$, show that the spectrum of $y(t)$ is limited to $-2W \leq \omega \leq 2W$.

Problem 1.8 (Optional)

(a) Find the Fourier transform of the *Gaussian* signal

$$x(t) = Ae^{-t^2/2T^2} \ , \quad -\infty < t < +\infty \ .$$

(b) If this signal goes through an LTI system with frequency response

$$H(\omega) = e^{-\frac{\omega}{2B^2}} \;, \quad -\infty < \omega < +\infty \;,$$

what is the output $y(t)$? Simplify the expression for $y(t)$ for $T >> 1/B$.

Problem 1.9

(a) Show that, if signal $g(t)$ has Fourier transform $G(\omega)$, then the Fourier transform of

$$g'(t) = g(t+T) + g(t-T)$$

is given by

$$G'(\omega) = 2G(\omega)\cos(\omega T) \;.$$

(b) What is the Fourier transform $G(\omega)$ of the signal

$$g(t) = \text{rect}(t-4) + \text{rect}(t+4) \;?$$

Note that the function $\text{rect}(t)$ is defined in the textbook as

$$\text{rect}(t) = \begin{cases} 1, & \text{if } |t| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) What is the signal $g(t)$ that has Fourier transform

$$G(\omega) = \text{rect}(\omega - 4) + \text{rect}(\omega + 4) \;?$$

Problem 1.10

Consider the following scenario:

- Signal $g_1(t) = 10^3\text{rect}(10^4 t)$ is applied as input to an ideal low-pass filter with frequency response $H_1(\omega) = \text{rect}(\omega/20000)$ to produce output $y_1(t)$.
- Signal $g_2(t) = \delta(t)$ is applied as input to an ideal low-pass filter with frequency response $H_2(\omega) = \text{rect}(\omega/10000)$ to produce output $y_2(t)$.
- Outputs $y_1(t)$ and $y_2(t)$ are then multiplied to obtain the final output $y(t) = y_1(t)y_2(t)$.

(a) Find $G_1(\omega)$ and $G_2(\omega)$.

(b) Find $h_1(t)$ and $h_2(t)$.

(c) Find $Y_1(\omega)$ and $Y_2(\omega)$; also, find the bandwidths of $y_1(t)$, $y_2(t)$ and $y(t)$.

Problem 1.11 (Optional)

A linear time-invariant system S is known to be stable. Show that, if the input $x(t)$ of S has finite energy, then the output $y(t)$ also has finite energy. In other words, show that if

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty \ ,$$

then

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt < \infty \ .$$

Problem 1.12 (Optional)

Show that $\int_{-\infty}^{+\infty} \text{sinc}^2(kx)dx = \pi/k$.

Problem 1.13

(a) Show that the Hilbert transform of an odd signal is an even function and that the Hilbert transform of an even signal is odd.

(b) Show that the energy content of a signal is equal to the energy content of its Hilbert transform.