

Problem Set 5

Probability Review, Hypothesis Testing, Detection and Estimation

Issued: Thursday, Oct. 13th.

Due: Thursday, Oct. 20th (beginning of lecture).

Reading from Lathi: Chapter 10.

Problem 5.1

Problems 10.1-2 and 10.1-11 from Lathi, pp. 480–481.

Problem 5.2

Problems 10.1-15 and 10.1-16 from Lathi, p. 482.

Problem 5.3

A communication system transmits symbols labeled 1, 2, and 3. The joint probability that symbol t is sent *and* symbol r is received is given in the following table for each pair (t, r) of sent and received symbols. For example, the probability that a 1 is sent and a 3 is received (due to noise in the channel/system) is given by 0.10 .

t	r		
	1	2	3
1	0.13	0.05	0.10
2	0.08	0.16	0.05
3	0.10	0.13	0.20

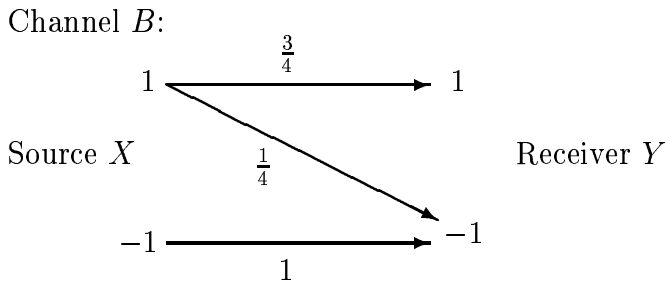
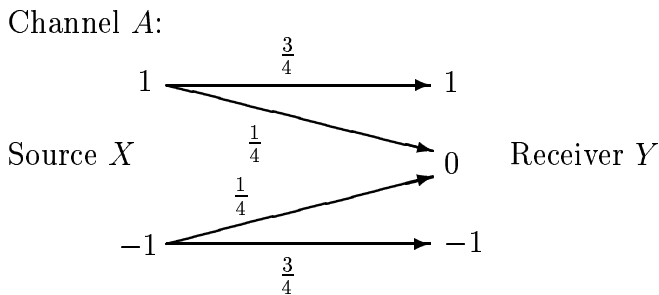
- Calculate the probability that symbol k was sent, given that symbol k was received (for $k = 1, 2, 3$).
- Find the decoding rule at the receiving end that minimizes the probability of error and calculate the associated probability of error (a decoding rule is a rule that maps the received symbol r to a transmitted symbol t).

Problem 5.4 (Optional)

Problems 10.5-2 and 10.5-3 from Lathi, p. 486.

Problem 5.5

A particular “antipodal” *pulse amplitude modulation* (PAM) communication system has a transmitter (“source”) that transmits the symbol $X = -1$ and the symbol $X = +1$ with equal probability. This source can use one of two channels, Channel A and Channel B . The figures below show the characteristics of each of these channels (the number next to an arrow denotes the probability of receiving the symbol at the right of the arrow, given transmission of the symbol at the left of the arrow). Channel A occasionally loses the symbol completely, while for Channel B the symbol $X = 1$ is at times misinterpreted as a -1 at the receiver. The receiver does not know explicitly which channel was used.



- If the source uses Channel A , what is the probability that $Y = -1$ is received? What are the probabilities of receiving $Y = 0$ and $Y = 1$? Repeat your calculations for the case when Channel B is used.
- Assume now that the source uses Channel A with probability α and Channel B with probability $1 - \alpha$. Suppose that the received symbol is $Y = -1$. What is the probability that Channel A was used, given that a $Y = -1$ was received?
- Suppose that you use the *maximum a posteriori probability* (MAP) rule to decide whether Channel A or Channel B was used. In other words, if $Y = -1$, you decide in favor of Channel A if

$$\Pr(A \text{ used} \mid Y = -1) > \Pr(B \text{ used} \mid Y = -1) .$$

Similarly, if $Y = 0$, you decide in favor of Channel A if

$$\Pr(A \text{ used} \mid Y = 0) > \Pr(B \text{ used} \mid Y = 0) ,$$

and, if $Y = 1$, you decided in favor of Channel A if

$$\Pr(A \text{ used} \mid Y = 1) > \Pr(B \text{ used} \mid Y = 1) .$$

What does the MAP rule reduce to in this case? For what range of α will you decide that Channel A was used, regardless of the value of Y that is received?

Problem 5.6

A Gaussian random variable X with zero mean and variance σ_X^2 is transformed to random variable Y via the transformation

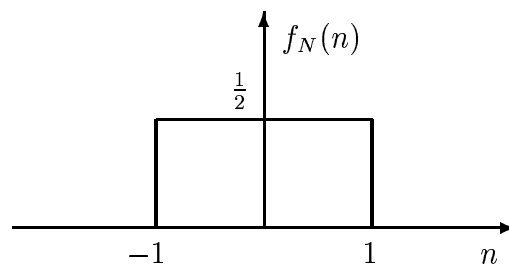
$$Y = X^2 .$$

Show that the probability density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y} \sigma_X} \exp\left(-\frac{y}{2\sigma_X^2}\right) , & y \geq 0 \\ 0 , & y < 0 . \end{cases}$$

Problem 5.7

Consider the following communication scenario: under hypothesis H_0 no signal is transmitted and the receiver observes a (constant) signal $R = N$, whereas under hypothesis H_1 a constant signal is transmitted and the receiver observes $R = 1 + N$. The random variable N models noise in the channel and is assumed to have the pdf shown below.



The prior probabilities for hypotheses H_0 and H_1 are given by

$$\Pr(H_0) = 1/4 , \quad \Pr(H_1) = 3/4 .$$

As we discussed in class, given a received signal $R = r$, we can use the MAP rule at the receiving end in order to minimize the probability of error. Furthermore, we know that the MAP rule reduces to the likelihood ratio test

$$\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{H_0}{\overset{H_1}{>}} \frac{\Pr(H_0)}{\Pr(H_1)} .$$

- (a) Find the range or ranges of values of the observation r for which you would decide that the signal was transmitted (“ H_1 ” has taken place).
- (b) Find the corresponding (minimal) probability of error.

Problem 5.8

Consider a system for determining whether a certain communication channel is being used or not. Let H_1 denote the hypothesis that the channel is being used, and let H_0 denote the hypothesis that the channel is not being used. The decision between hypotheses H_1 and H_0 is based on a single scalar measurement R (e.g., the output of our antenna at a particular time). This measurement R is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 : f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 : f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} .$$

Assume that the a priori probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

- (a) Find the decision rule that minimizes the probability of error.
- (b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 ,$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma_1$ and γ_2 are appropriate constants, and $Q(\gamma)$ is defined as

$$Q(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau .$$

- (c) Is there a choice of p_0 and p_1 such that the rule of part (b) always decides in favor of one of the hypothesis regardless of the measurement R ?

Problem 5.9

The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} A(1 - |x - y|), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find A .
- (b) **(optional)** Find the marginal pdf of X and Y .
- (c) Find $\Pr(X + Y < 1 \mid X > \frac{1}{2})$.