University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2005

Problem Set 6

MMSE Estimation, Linear MMSE Estimation, Jointly Gaussian R.V.'s

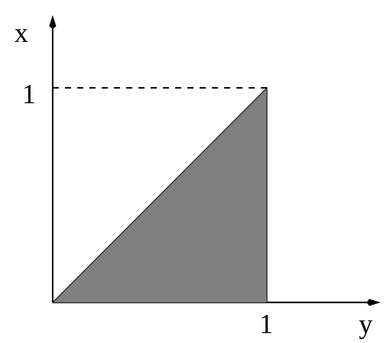
Issued: Thursday, Oct. 20st. **Due:** Thursday, Oct. 27th (beginning of lecture).

Reading from Lathi: Chapter 10.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 10th, from 7:00pm to 9:00pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 3rd. The corresponding material includes Problem Sets 1 through 7 and Chapters 1, 2, 3, 4, 5, 10 and 11 (excluding Section 11.6) from Lathi. Emphasis will be placed on the material not covered in the first Mid-Semester Exam.
For the exam, you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 6.1

Random variables X and Y have joint pdf $f_{X,Y}(x, y)$ that is constant in the shaded region (and zero elsewhere).



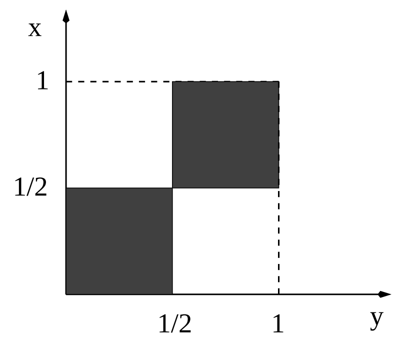
(a) Make fully labeled sketches of the densities $f_X(x)$ and $f_Y(y)$.

(b) Are X and Y statistically independent? Explain.

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

Problem 6.2

Random variables X and Y have joint pdf $f_{X,Y}(x, y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Are X and Y uncorrelated? Are X and Y statistically independent?

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

(d) Determine $\widehat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X, given the observation $Y = y$.

Problem 6.3

In a certain wireless communication system, the transmitted value X is attenuated by a random attenuation and is corrupted by channel noise so that the available measurement Y at the receiving end is related to X as

$$Y = WX + N \ .$$

The transmitted value X is a uniform random variable in the interval $[-1,\ 1]$, the attenuation W is a uniform random variable in the interval $[\frac{1}{2},\ 1]$, and the additive noise N is a Gaussian random variable with zero mean and unit variance. Furthermore, X, W and N are mutually independent.

Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

Hint: The following may make your calculation easier: $E[X] = 0$, $E[X^2] = 1/3$, $E[W] = 3/4$, $E[W^2] = 7/12$.

Problem 6.4

A signal X which has Gaussian distribution with mean $\mu_X = 2$ and variance $\sigma_X^2 = 0.5$ is sent via a cable. The receiver needs to form an estimate $\widehat{X}$ for the transmitted signal X based on the received signal $Y = X + N$, where N models the (*additive*) noise. Specifically, N is a Gaussian random variable that is independent of X and has mean $\mu_N = 0$ and variance $\sigma_N^2 = 2$. Find $\widehat{X}_{\text{MMSE}}$, the minimum mean square error estimator for X given Y.