

Problem Set 9
Signal Detection in Noise, Matched Filtering

Issued: Thursday, December 1st.

Due: Thursday, Dec. 8th (beginning of lecture).

Reading from Lathi: Chapter 13, Sections 13.1–13.2.

Announcement: The Final Exam will be held on Saturday, December 17th, from 1:30pm to 4:30pm in 165 Everitt Laboratory. The exam will cover all material from the beginning of the term. For the exam, you can bring *three* 8.5 × 11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 9.1

Consider the signal

$$p(t) = \begin{cases} 2A\frac{t}{T}, & 0 \leq t \leq T/2, \\ -2A(1 - \frac{t}{T}), & T/2 \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

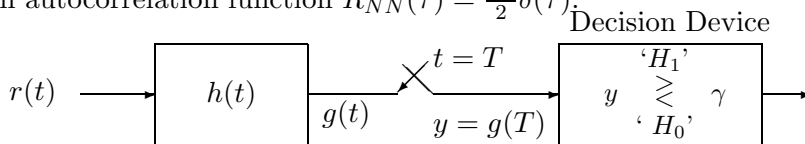
- (a) Sketch the impulse response of a filter that is matched to this signal.
- (b) Plot the output of the matched filter in part (a) when $p(t)$ is the input. What is the maximum value of the output?

Problem 9.2

In a binary on-off signaling scheme the transmitted signal $s(t)$ is either set to zero (under hypothesis H_1), or is given by

$$p(t) = \begin{cases} At, & 0 \leq t \leq T, \\ 0, & \text{otherwise} \end{cases}$$

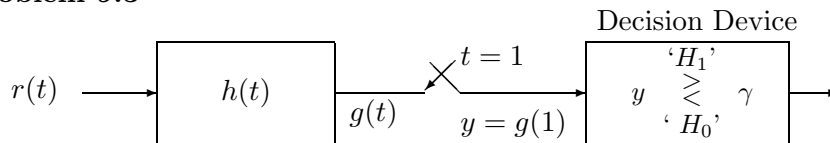
(under hypothesis H_0). The *a priori* probabilities for the two hypotheses are given by $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$. The signal $r(t)$ seen at the receiving end includes additive noise, i.e., $r(t) = s(t) + n(t)$, where $n(t)$ is a sample function of a white Gaussian random process $N(t)$ with autocorrelation function $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$.



In order to make a decision as to whether hypothesis H_0 or H_1 took place we use the approach shown below. The impulse response $h(t)$ is given by $h(t) = 1$ for $0 \leq t \leq T$ (0 otherwise). At the decision device, the value y observed for random variable Y (i.e., the value of $g(t)$ at time $t = T$) is compared to a threshold γ . The decision is “ H_0 ” if $y > \gamma$, and “ H_1 ” otherwise.

- (a) Find $f_{Y|H_0}(y|H_0)$, the conditional probability density of Y given hypothesis H_0 .
- (b) Choose γ so that the probability of error is minimized.

Problem 9.3



Consider the binary hypothesis testing situation shown above. Signal $r(t) = s(t) + n(t)$, where $s(t)$ is a deterministic signal and $n(t)$ is noise. Under hypothesis H_0 , signal $s(t)$ is given by

$$p(t) = \begin{cases} \sin(\frac{\pi}{2}t), & 0 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Under hypothesis H_1 , signal $s(t)$ is zero. Assume that the *a priori* probabilities for the two hypotheses are equal and that the noise $n(t)$ is a sample function of a white Gaussian random process $N(t)$ with average power $N_0/2$. At the decision device, the value y observed for random variable Y is compared to a threshold γ . The decision is “ H_0 ” if $y > \gamma$, and “ H_1 ” otherwise.

- (a) Let $T = 1$ and $h(t)$ be given by

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{Y|H_0}(y|H_0)$ and $f_{Y|H_1}(y|H_1)$, the conditional probability densities of Y given each of the two hypotheses. Choose γ to minimize the probability of error and find the corresponding probability of error.

- (b) Repeat part (a) for $T = 2$.
- (c) Let $T = 2$. Choose $h(t)$ and γ so that you minimize the probability of error. What is the corresponding probability of error?

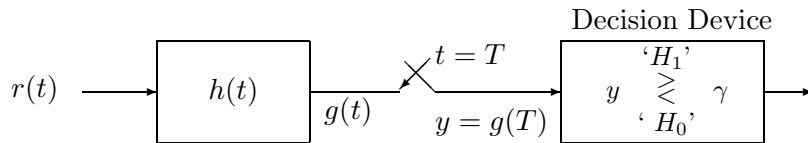
Problem 9.4

Consider the following communication scenario: the transmitter sends one bit (“0” or “1”) by setting the transmitted signal $s(t)$ either to zero or to the signal $p(t) = e^{-t}u(t)$; the signal $r(t)$

at the receiver is corrupted by additive noise, i.e., $r(t) = s(t) + n(t)$, where $n(t)$ is a sample path of a white Gaussian random process $N(t)$ with power spectral density $S_{NN}(\omega) = \frac{N_0}{2}$. The receiver is faced with the following binary hypothesis testing problem:

$$\begin{aligned} \text{Hypothesis } H_0 \text{ ("0" being transmitted)} & : r(t) = n(t), \\ \text{Hypothesis } H_1 \text{ ("1" being transmitted)} & : r(t) = p(t) + n(t). \end{aligned}$$

The *a priori* probabilities for the two hypotheses are equal. The receiver needs to make a decision as to whether hypothesis H_0 or H_1 took place with finite delay. It uses the system shown below where the constant T is to be specified. The impulse response of the filter $h(t)$ and threshold γ will be chosen appropriately.



(a) Assume that $T = 1$ and that $h(t)$ is given by

$$h(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (i) Find $f_{Y|H_0}(y|H_0)$ and $f_{Y|H_1}(y|H_1)$, the conditional probability densities of the random variable $Y = g(T)$ under hypotheses H_0 and H_1 .
- (ii) Choose γ so that you minimize the probability of error. Express the corresponding probability of error in terms of the Q -function.

(b) Repeat Part A for $T = 1$ and $h(t) = p(1-t)u(t)$ where $u(t)$ is the unit step function.

(c) If $T = 1$ and $h(t)$ has to be causal, is there a better choice for $h(t)$ than the choice in Part B? If so, specify $h(t)$ and calculate γ and the probability of error in terms of the Q -function. If not, justify your answer.

Problem 9.5

A transmitter transmits one bit ("0" or "1") by setting the transmitted signal $s(t)$ either to zero or the pulse $p(t)$ shown below.

$$p(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

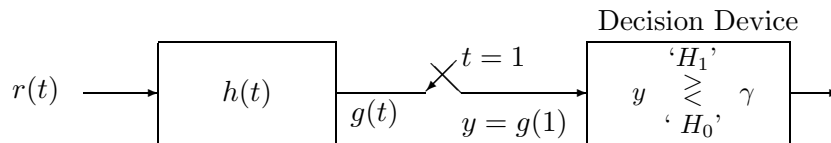
The signal $r(t)$ received at the receiver is corrupted by additive noise, i.e., $r(t) = s(t) + n(t)$, where $n(t)$ is a sample path of a (*non-white*) Gaussian random process $N(t)$ with zero mean and autocorrelation function $R_{NN}(\tau)$ shown below.

$$R_{NN}(\tau) = \begin{cases} 2(2 - \tau), & 0 \leq \tau \leq 2, \\ 2(2 + \tau), & -2 \leq \tau \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The receiver is faced with the following binary hypothesis testing problem:

$$\begin{aligned} \text{Hypothesis } H_0 \text{ ("0" being transmitted)} & : r(t) = n(t), \\ \text{Hypothesis } H_1 \text{ ("1" being transmitted)} & : r(t) = p(t) + n(t). \end{aligned}$$

The *a priori* probabilities for the two hypotheses are unequal with $\Pr(H_0) = 2\Pr(H_1)$. In order to make a decision as to whether hypothesis H_0 or H_1 took place, the receiver uses the system shown below.



We will analyze the performance of this detection scheme for the following receiver filter $h(t)$:

$$h(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find $f_{Y|H_0}(y|H_0)$, the conditional probability density of Y under hypothesis H_0 .
- Find $f_{Y|H_1}(y|H_1)$, the conditional probability density of Y under hypothesis H_1 .
- Choose the threshold γ so that the probability of error is minimized.