University of Illinois at Urbana-Champaign

ECE 434: Random Processes

Spring 2003
Final Exam

Monday, May 12, 2003

Name: ____

- You have three hours for this exam. The exam is closed book and closed note, except that you may consult both sides of three sheets of notes, typed in font size 10 or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (21 pts.)

2. __________ (12 pts.)

3. __________ (12 pts.)

4. __________ (7 pts.)

5. __________ (6 pts.)

6. __________ (10 pts.)

7. __________ (12 pts.)

Total: __________(80 pts.)

Problem 1 *(21 points)* Indicate true or false for each statement below and *justify your answers.* (One third credit is assigned for correct true/false answer without correct justification.)

(a) If $\mathcal{H}(z)$ is a positive type z-transform, then so is $\cosh(\mathcal{H}(z))$. (Recall that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.)

(b) If X is a m.s. differentiable stationary Gaussian random process, then for t fixed, X_t is independent of the derivative at time t: X'_t.

(c) If $X = (X_t : t \in \mathbb{R})$ is a WSS, m.s. differentiable, mean zero random process, then X is mean ergodic in the mean square sense.

(d) If $M = (M_t : t \geq 0)$ is a martingale with $E[M_t^2] < \infty$ for each t, then $E[M_t^2]$ is increasing in t (i.e. $E[M_s^2] \leq E[M_t^2]$ whenever $s \leq t$.)

(e) If $X_1, X_2, \ldots,$ is a sequence of independent, exponentially distributed random variables with mean one, then there is a finite constant K such that $P[X_1 + \cdots + X_n \geq 2n] \leq K \exp(-n^2)$ for all n.

(f) If X and Y are random variables such that $E[X^2] < \infty$ and $E[X|Y] = Y$, then
$E[X^2|Y^2] = E[X|Y]^2$.

(g) If $N = (N_t : t \geq 0)$ is a random process with $E[N_t] = \lambda t$ and $E[N_s N_t] = \lambda \min\{s, t\}$ for $s, t \geq 0$, then N is a Poisson process.

Problem 2 *(12 points)* Let N be a Poisson random process with rate $\lambda > 0$ and let $Y_t = \int_0^t N_s ds$.
(a) Sketch a typical sample path of Y and find $E[Y_t]$.

(b) Is Y m.s. differentiable? Justify your answer.

(c) Is Y Markov ? Justify your answer.

(d) Is Y a martingale? Justify your answer.

Problem 3 *(12 points)* Let $X_t = U\sqrt{2}\cos(2\pi t) + V\sqrt{2}\sin(2\pi t)$ for $0 \le t \le 1$, where U and V are independent, $N(0,1)$ random variables, and let $N = (N_\tau : 0 \le \tau \le 1)$ denote a real-valued Gaussian white noise process with $R_N(\tau) = \sigma^2\delta(\tau)$ for some $\sigma^2 \ge 0$. Suppose X and N are independent. Let $Y = (Y_t = X_t + N_t : 0 \le t \le 1)$. Think of X as a signal, N as noise, and Y as an observation.
(a) Describe the Karhunen-Loève expansion of X. In particular, identify the nonzero eigenvalue(s) and the corresponding eigenfunctions.

There is a complete orthonormal basis of functions $(\phi_n : n \ge 1)$ which includes the eigenfunctions found in part (a) (the particular choice is not important here), and the Karhunen-Loève expansions of N and Y can be given using such basis. Let $\tilde{N}_i = (N, \phi_i) = \int_0^1 N_t \phi_i(t)dt$ denote the i^{th} coordinate of N. The coordinates $(\tilde{N}_1, \tilde{N}_2, \ldots)$ are $N(0, \sigma^2)$ random variables and $U, V, \tilde{N}_1, \tilde{N}_2, \ldots$ are independent. Consider the Karhunen-Loève expansion of Y, using the same orthonormal basis.
(b) Express the coordinates of Y in terms of $U, V, \tilde{N}_1, \tilde{N}_2, \ldots$ and identify the corresponding eigenvalues (i.e. the eigenvalues of $(R_Y(s,t) : 0 \le s, t \le 1)$).

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(c) Describe the minimum mean square error estimator $\widehat{U}$ of U given $Y = (Y_t : 0 \leq t \leq 1)$, and find the minimum mean square error. Use the fact that observing Y is equivalent to observing the random coordinates appearing in the KL expansion of Y.

Problem 4 *(7 points)* Let $(X_k : k \in \mathbb{Z})$ be a stationary discrete-time Markov process with state space $\{0, 1\}$ and one-step transition probability matrix $P = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. Let $Y = (Y_t : t \in \mathbb{R})$ be defined by $Y_t = X_0 + (t \times X_1)$.

(a) Find the mean and covariance functions of Y.

(b) Find $P[Y_5 \geq 3]$.

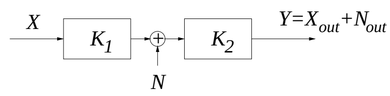
Problem 5 *(6 points)* Let Z be a Gauss-Markov process with mean zero and autocorrelation function $R_Z(\tau) = e^{-|\tau|}$. Find $P[Z_2 \geq 1 + Z_1 | Z_1 = 2, Z_0 = 0]$.

Problem 6 *(10 points)* Let X be a real-valued, mean zero stationary Gaussian process with $R_X(\tau) = e^{-|\tau|}$. Let $a > 0$. Suppose X_0 is estimated by $\widehat{X}_0 = c_1 X_{-a} + c_2 X_a$ where the constants c_1 and c_2 are chosen to minimize the mean square error (MSE).

(a) Use the orthogonality principle to find c_1, c_2, and the resulting minimum MSE, $E[(X_0 - \widehat{X}_0)^2]$. (Your answers should depend only on a.)

(b) Use the orthogonality principle again to show that $\widehat{X}_0$ as defined above is the minimum MSE estimator of X_0 given $(X_s : |s| \geq a)$. (This implies that X has a two-sided Markov property.)

Problem 7 *(12 points)*
Suppose X and N are jointly WSS, mean zero, continuous time random processes with $R_{XN} \equiv 0$. The processes are the inputs to a system with the block diagram shown, for some transfer functions $K_1(\omega)$ and $K_2(\omega)$:



Suppose that for every value of ω, $K_i(\omega) \neq 0$ for $i = 1$ and $i = 2$. Because the two subsystems are linear, we can view the output process Y as the sum of two processes, X_{out}, due to the input X, plus N_{out}, due to the input N. Your answers to the first four parts should be expressed in terms of K_1, K_2, and the power spectral densities S_X and S_N.
(a) What is the power spectral density S_Y?

(b) What is the signal-to-noise ratio at the output (equal to the power of X_{out} divided by the power of N_{out})?

(c) Suppose Y is passed into a linear system with transfer function H, designed so that the output at time t is $\widehat{X}_t$, the best (not necessarily causal) linear estimator of X_t given $(Y_s : s \in \mathbb{R})$. Find H.

(d) Find the resulting minimum mean square error.

(e) The correct answer to part (d) (the minimum MSE) does not depend on the filter K_2. Why?