University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 434: Random Processes

Spring 2004

Problem Set 2

Probability Review, Sequences of Random Variables

Issued: Monday, Feb. 9th **Due:** Beginning of lecture on Monday, Feb. 23rd

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Reading from Hajek: Chapter 2.

Reading from Stark and Woods: Chapter 4; Chapter 6, Sections 6.7 and 6.8.

Problem 2.1

Given a random variable X, prove the following identity:

$$E[(X-c)^2] = \mathrm{Var}(X) + (c - E[X])^2, \quad \forall c \in \Re.$$

Hence if we would like to approximate X by a constant $c \in \Re$ such that the mean squared error $E[(X-c)^2]$ is minimized, the optimal choice is $c = E[X]$.

Problem 2.2

Let Θ be uniformly distributed on the interval $[0, 2\pi]$ and let $X_n = cos(n\theta)$. [**Hint:** $cos(n\theta)cos(m\theta) = (cos((n-m)\theta) + cos((n+m)\theta))/2$.]

(a) Does X_n converge almost surely as n tends to infinity? Justify your answer.

(b) Does X_n converge in mean square sense as n tends to infinity? Justify your answer.

(c) Does X_n converge in distribution as n tends to infinity? Justify your answer.

Problem 2.3

From Hajek, Chapter 2: Problem 2. Also answer the following part (c): $X_n(\omega) = \sqrt{n}\omega$ if $0 \leq \omega \leq 1/n$ and $X_n(\omega) = 0$ otherwise.

Problem 2.4

From Hajek, Chapter 2: Problem 3. Also answer the following part (d): In what sense(s) does the sequence (X_n) converge as n tends to infinity?

Problem 2.5

Given a sequence of random variables (X_n), prove the following statements.

(a) Convergence in mean of order $p > 0$ implies convergence in probability.

(b) If $\lim_{n\to\infty} X_n \to X(\text{p.})$ and $\lim_{n\to\infty} X_n \to Y(\text{p.})$, then $P(X \neq Y) = 0$, i.e., $X = Y$ (a.s.).

(c) If $\lim_{n\to\infty}(X_n - X)^2 = 0$ (a.s.), then $\lim_{n\to\infty} X_n^2 = X^2$ (a.s.).

Problem 2.6

From Hajek, Chapter 2: Problems 4, 5 and 6.

Problem 2.7

From Hajek, Chapter 2: Problems 7, 8 and 9.

Problem 2.8

While deciding upon which textbook to use for ECE434, we considered several options. One potential textbook, by Author P., decided to depart from the usual definition of the Cauchy criterion for convergence and offered instead the following definition, which we will call the P-Cauchy criterion:
Non-Standard Cauchy Criterion(a.k.a. P-Cauchy criterion): A sequence x_n is a P-Cauchy sequence if and only if for every integer $m > 0$, $\lim_{n\to\infty} |x_n - x_{n+m}| = 0$.

(a) Consider the sequence of i.i.d. random variables y_i such that

$$P(y_i = 1) = P(y_i = 0) = 0.5\ .$$

Let $x_n = \sum_{k=1}^{n}(y_k/k)$. Use the standard definition of the Cauchy criterion to determine whether or not this sequence converges in the mean-square sense.

(b) Using the non-standard P-Cauchy criterion, Author P. would have us formulate mean-square convergence of x_n as the following test: sequence x_n converges in the mean-square sense if and only if

$$\lim_{n\to\infty} E((x_n - x_{n+m})^2) = 0$$

for every integer $m \geq 0$. Does the above limit exist? Using this convergence test, what conclusions would you have drawn about the sequence x_n? Does this match your answer to the previous part of this problem?

Problem 2.9 (optional)

From Hajek, Chapter 2, Problems 10, 11 and 12.