Optimal Control
A Game Theoretic Approach

GE 493 – Game Theory

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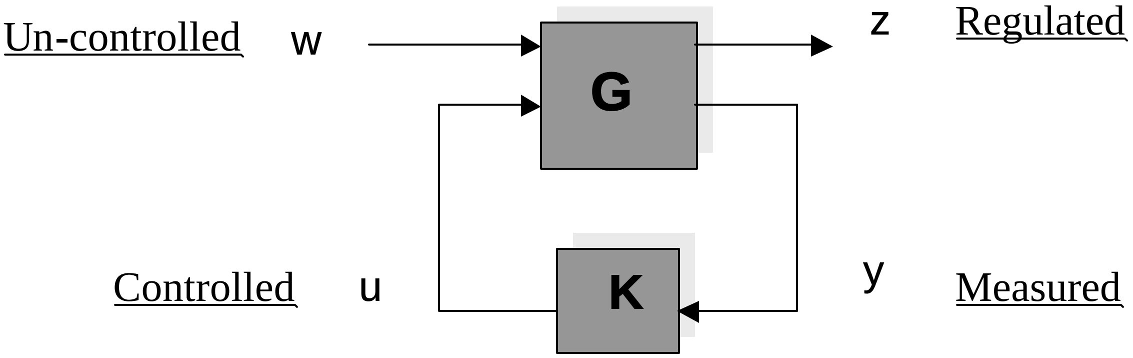
Robust Control



Robustness

- Stability (stable under perturbations from nominal model)
- Performance (perform satisfactorily under perturbations).

Aim

- Disturbance Rejection/attenuation
- Design u so as to keep z “small”

Game Theoretic Approach

Problem is viewed as a Game between Man (Controller) and Nature (unknown disturbances).

Controller Design Problem

State Equations

$$\dot{x} = A(t)x(t) + B(t)u(t) + D(t)w(t), \quad x(0) = 0$$

$$u(t) = m(t, x_{[0,t_f]}), \qquad t \in [0,t]$$

Player 1: **Controller**

$m \in \mathrm{M}_{\mathrm{cl}}$ *all closed loop mappings*

Player 2: **Disturbance**

$$w \in H_w$$

Cost Function

$$L(u,w) = \left| x(t_f) \right|^2_{Q_f} + \int_0^{t_f} \left\{ \left| x(t) \right|^2_{Q_f} + \left| u(t) \right|^2 \right\} dt$$

Linear Quadratic Dynamic Game

Saddle Point

Design a controller such that the gain from w to z is minimized

Minimize:

$$\inf_{m\in M} \left\langle\left\langle T_m \right\rangle\right\rangle =: g^*$$

where

$$\left\langle\left\langle T_m \right\rangle\right\rangle := \sup_{w\in H_w} \left\| T_m(w) \right\|_z \Big/ \left\| w \right\|_w$$

$$L_g(m,w) = \left\| T_m(w) \right\|_z^2 - g^2 \left\| w \right\|_w^2 \equiv \tilde{L}(m,w) - g^2 \left\| w \right\|_w^2$$

Find smallest value of γ > 0 such that upper value of the game (with objective function L_γ) is bounded

Saddle Point Condition

$$L_g(m^*,w) \le L_g(\underbrace{m^*,w^*}_{\text{SaddlePoint}}) \le L_g(m,w^*), \qquad \forall m \in \mathbf{M}, w \in H_w$$

$$\overbrace{\inf_{\grave{\imath}\in M} \sup_{w\in H_w} \left\| T_m(w) \right\|_z \Big/ \left\| w \right\|_w}^{Upper\,Value} \ge \underbrace{\sup_{w\in H_w} \inf_{\grave{\imath}\in M} \left\| T_m(w) \right\|_z \Big/ \left\| w \right\|_w}_{Lower\ \ Value}$$

Saddle Point Solution

- Introduce a matrix differential equation (Riccati) associated with the game.

- Game admits a unique strongly time consistent saddle-point policy m* iff the M.D.E does not have a conjugate point in $[0,t_f]$.

- <u>If above condition is not satisfied</u>

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No saddle point

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disturbance will drive L_γ arbitrarily large !!

Measurement Schemes

- Perfect State Measurement.

$$u(t) = m(t; x(s), s \leq t), \qquad t \geq 0$$

- Delayed State Measurement.

$$u(t) = m(t; x_{0,t-q\,]}), \; t \geq q$$

- Sampled State Measurement.

$$u(t) = m(t; x(t_k),.........,x(t_1), x_0), \quad t_k \leq t \leq t_{k+1}$$

Robustness to Plant Perturbations

System Equations

$$\dot{x} = (A+E)x + Bu, \qquad x(0) = x_0 \in \mathfrak{R}^n$$

Plant perturbation

Measurement $y = Cx$

Control $u = Ky$

$$E = \sum_{i=1}^{n} G_i L_i H_i \qquad tr(L_i^T L_i) \le d_i^2$$

Want to stabilize the system for all possible E.

Cost Function

$$J = \int_0^\infty (x^T Q x + u^T R u)dt$$

Find K^* *and* E^* such that :

$$J(K^*, E^*) = \max_E J(K^*, E) = \inf_K \max_E J(K, E)$$

Assumption:

The optimizing solution E^* lies on the boundary

Assumptions & Solution

- Instead of the hard-constrained problem (robust control), an alternate soft-constrained (dynamic game) problem is solved.

- The assumption is that the solution to the hard problem lies on the boundary.

$$\dot{x} = Ax + Bu + \sum_{i=1}^{n} G_i w_i \qquad z_i = H_i x$$

$$J_g \equiv J - \sum_{i=1}^{n} g_i tr(L_i^T L_i) \qquad w_i = L_i z_i$$

No constraints on L_i

- Solution to the 2nd problem is the the solution to the original problem (under the above assumptions)

Conclusion and Discussion

- Two different robustness issues were dealt with using a Game Theoretic approach.
- Both problems were analyzed in a Quadratic dynamic game framework.
- Different measurement schemes made the analysis more practical (real-life).
- The assumption that the solution exists on the boundary was not justified (no practical examples were provided).